Wave chaotic analysis of weakly coupled reverberation chambers

Gabriele Gradoni #1, Jen-Hao Yeh #2, Thomas M. Antonsen Jr. #3, Steven Anlage #4, and Edward Ott #5

 # Institute for Research in Electronics and Applied Physics
University of Maryland, College Park MD 20742, USA
1 gabriele.gradoni@gmail.com 2 davidyeh@umd.edu 3 antonsen@umd.edu 4 anlage@umd.edu 5 edott@umd.edu

Abstract—In this paper, we analyze the field fluctuations in *weakly coupled* complex cavities by using a random matrix theory to model the chaotic scattering within each cavity. Universal (chaotic) and non-universal are separated. In particular, non-universal are found to be conveniently described by the radiation impedance concept. Inherently, the development of the random field regime is accounted for by taking each mode of the cavity as a random plane wave expansion. Sources and sinks inside the cavities are assumed to be electrically small. A model for the cascaded cavities scenario is derived through the electric network theory and random matrix theory for both lossy and lossless cases. The adopted physical framework is a linear chain of two-port cavities terminated by a one-port cavity. The field flowing into this last cavity is related to the current excitation on the first cavity through the *coupling radiation impedance*. Closed-form expressions are derived for two interconnected cavities, mimicking the nested reverberation chamber scenario. Finally, the practical issue of measurements in a nested reverberation chamber is presented and discussed. Accordingly, based on physical arguments, the small fluctuations theory applies. Results are of interest in interference propagation through complex electromagnetic environment or planar circuits, EMC immunity tests, and reverberation chambers.

I. INTRODUCTION

Over the last decades, there has been an increasing interest in coupling of complex systems.
Several studies on the field properties inside electromagnetic complex cavities have been performed in both the electromagnetic compatibility (EMC) and the theoretical physics communities. The connection between reverberation chambers and chaotic cavities has been investigated [1], [2]. Modeling of reverberation chambers is grounded on the random plane-wave spectrum, and statistical properties of the total field and power/intensity are derived by taking ideal random complex fields. On the other hand, by using random matrix theory (RMT), the energy spectrum of chaotic cavities is assumed to be discrete and uniformly distributed, so as the local field can be expanded in a series of eigenfunctions with known statistics. In the semiclassical limit, each cavity eigenmode (wavefunction) can be *locally* represented by a plane-wave superposition (Berry's conjecture) [3], [4], [5]. Starting form Maxwell's equation for a quasi-planar cavity, upon application of RMT and Berry's conjecture, one arrives to a closed-form solution of the cavity impedance matrix, relating port voltages to currents [6], [7].

In this paper, we derive closed-form expressions for the coupling impedance of interconnected cavities. We assume that the coupling between two contiguous cavities takes place through a small leaking port. Furthermore, the electromagnetic energy flows from one cavity to another by an electrically short cable, thus modeled as a direct connection. The coupling impedance is defined as a complex scalar, linearly relating the port voltage at the last cavity to the current excitation at the first port of the chain. Joint application of electrical network theory and random matrix theory (RMT) – i.e., the Random Coupling Model (RCM) – leads to a compact form for that impedance, where in the high-losses limit it is possible to separate the non-universal (average) contribution from the universal (fluctuating) part.
Such a model serves to investigate the physical properties and coupling transformations in chains of heterogeneous (mixed random, deterministic, chaotic or regular) EM cavities or structures. Results are of interest in weak coupling of nested reverberation chambers, transmission of interference in complex electromagnetic systems, coupling of complex electrical systems with several fluctuating elements, and chaotic chains in general.

II. RANDOM MATRIX THEORY MODEL FOR COUPLING

RMT can be used to describe the chaotic wave scattering inside a complex electromagnetic cavity [8]. Concerning the reverberation chamber, we conveniently describe the inner field by averaging over the statistical ensemble of cavities perturbed by different (statistically independent) configurations of mechanical mode-stirrers [3]. Accordingly, even though a full *vectorial* characterization of the local field would require the calculation of the cavity (dyadic) Green's function, the average field characterization can be performed considering considering the *scalar* Helmholtz equation.
A rigorous calculation has been previously carried out in terms of port voltages and currents, thus considering the cavity as a two-port network and exploiting the *Telegrapher's* equations. In such a scenario, the cavity radiation impedance has been calculated, relating voltages and currents between the ports in the absence of scattering from the walls of the enclosure [6], [7]. This captures the non-universal features associated with the ports.

The coupling between two cavities is assumed to take place through a concentrated (electrically small) perturbation; this allows for approximating sources and leaks with ports.

We assume that the first cavity is excited by a small (with respect to the cavity dimensions) antenna/source, and that the inner field is eventually picked up by a small antenna/sink in the last cavity. It is then straightforward to model this system as a two-port network, fully characterized by a 2×2 impedance matrix. The bridging between electrical network theory and field theory is carried out introducing the following port voltage definition [6]

$$V^{(c)} = \iint V_T^{(c)}(x,y)\, u(x,y)\ dx dy \ , \tag{1}$$

where the local voltage is related to the electric field by $V_T^{(c)}(x,y) = -E(x,y)\,h$ for the cavity c. Here, h is the height of the quasi-planar cavity, where the field is quasi-electrostatic. In a 3D reverberation chamber, assuming the *ensemble* field homogeneity and ergodicity [3], the *average* Cartesian field can be treated as a scalar field.

A. Two interconnected cavities

Starting from Maxwell's equations we write, for a two-port and a single-port cavity [6], [7]

$$\left(\nabla_\perp^2 + k_1^2\right)\hat{V}_T^{(1)} = -jk_1 h_1 \eta_0 \sum_{i=1}^{2} u_i^{(1)}\hat{I}_i^{(1)} \ , \tag{2}$$

$$\left(\nabla_\perp^2 + k_2^2\right)\hat{V}_T^{(2)} = -jk_2 h_2 \eta_0 u_1^{(2)}\hat{I}_1^{(2)} \ . \tag{3}$$

Following the usual approach, we find a solution to the wave equations expanding the voltage in eigenfunctions, hence

$$\hat{V}_T^{(1)} = \sum_{n1} c_{n1}\phi_{n1} \ , \tag{4}$$

$$\hat{V}_T^{(2)} = \sum_{n2} c_{n2}\phi_{n2} \ , \tag{5}$$

where ϕ_n are the voltage eigenfunctions, and with coefficients

$$c_{n1} = \frac{-jk_1 h_1 \eta_0 \sum_{i=1}^{2}\left\langle u_i^{(1)}\phi_{n1}^{(1)}\right\rangle \hat{I}_i^{(1)}}{k_1^2 - k_{n1}^2} \ , \tag{6}$$

$$c_{n2} = \frac{-jk_2 h_2 \eta_0 \left\langle u_1^{(2)}\phi_{n2}^{(2)}\right\rangle \hat{I}_1^{(2)}}{k_2^2 - k_{n2}^2} \ , \tag{7}$$

found upon multiplying (6) and (7) by ϕ_{n1} and ϕ_{n2} respectively, and integrating over the chamber volume. Two weakly coupled chaotic cavities can thus be modeled by direct interconnection of double- and single-channel networks. This means that, since the coupling takes place through a small aperture or a material sample, the energy leakage can be related to a single mode of propagation, whence the transmission line approximation becomes valid. According to the electrical network theory, the first cavity is described by two constitutive equations

$$V_1^{(1)} = Z_{11}^{(1)}I_1^{(1)} + Z_{12}^{(1)}I_2^{(1)} \ , \tag{8}$$

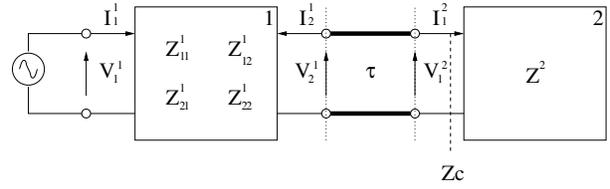$$V_2^{(1)} = Z_{21}^{(1)}I_1^{(1)} + Z_{22}^{(1)}I_2^{(1)} \ , \tag{9}$$

Fig. 1. Network model of two weakly *coupled* reverberation chambers: the first cavity is excited by a strongly localized current distribution, while the power leaking out excites the second one. The electrical network theory arises when $\tau \to 0$.

which can be gathered into the following matrix equation

$$V_1^{(2)} = Z^{(2)}I_1^{(2)} \ , \tag{10}$$

while the second (one-port) cavity is described by a scalar equation without crossing impedances

$$V_1^{(2)} = Z_{11}^{(2)}I_1^{(2)} \ . \tag{11}$$

The basic model of coupling, realized connecting two (adiabatic) random/chaotic cavities and exciting the first one by injection of an external current through a *localized* radiating element (antenna) inside the cavity, is depicted in Fig. 1. To assume direct interconnection between cavities does mean nodelay, i.e., $\tau \approx 0$. If some field distribution appears in the second cavity, this can be only due to the indirect excitation originated by the first cavity. The coupling is formally realized introducing the following boundary conditions at the interconnecting section

$$V_2^{(1)} = V_1^{(2)} \ , \tag{12}$$

$$I_2^{(1)} = -I_1^{(2)} \ , \tag{13}$$

in the constitutive equations (10) and (11), obtaining a coupling equation for currents

$$Z_{21}^{(1)}I_1^{(1)} + \left(Z_{22}^{(1)} + Z_{11}^{(2)}\right)I_2^{(1)} = 0 \ . \tag{14}$$

Upon direct substitution of $\hat{I}_2^{(1)}$ from (14) into (11), multiplying by $u^{(2)}$, integrating, and using the voltage boundary condition again yields

$$V_1^{(2)} = \sum_{n2} jk h_2 \eta_0 \frac{\left\langle u^{(2)}\phi_{n2}\right\rangle^2 I_2^{(1)}}{k_2^2 - k_{n2}^2} = -Z^{(2)}\left(\frac{V_2^{(2)} + Z_{21}^{(1)}I_1^{(1)}}{Z_{22}^{(1)}}\right) \ , \tag{15}$$

which relates the voltage induced in the second cavity to the current excitation of the first cavity, through the following *coupling impedance*

$$Z_{c,2} = \frac{Z^{(2)}Z_{21}^{(1)}}{Z^{(2)} + Z_{22}^{(1)}} \ , \tag{16}$$

formally identical to the cross-impedance Z_{21} of a cascade connection of two two-ports networks [9, see (27) p. 170].

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B. Lossy cavity: weak fluctuation approximation

The expression in (16) is exact and also valid for cavities with different geometry and electromagnetic parameters. For instance, in high lossy scenario, the reduction of the magnitude of fluctuation can be modeled by using first-order perturbation theory [10]. In our context, the derivation of each impedance involved in (16) is undertaken by RCM, thus they can be separated into a mean value $\langle Z \rangle$ plus a fluctuating part δZ, as usually done for the scattering matrix elements in quantum chaos. Upon inserting such a decomposition into (16) yields

$$Z_{c,2} \approx \frac{(\langle Z^{(2)} \rangle + \delta Z^{(2)}) \langle Z_{21}^{(1)} \rangle + \langle Z^{(2)} \rangle \delta Z_{21}^{(1)} + \delta Z^{(2)} \delta Z_{21}^{(1)}}{\langle Z^{(2)} \rangle + \delta Z^{(2)} + \langle Z_{22}^{(1)} \rangle + \delta Z_{22}^{(1)}} \tag{17}$$

The second-order fluctuation term $\delta Z^{(2)} \delta Z_{21}^{(1)}$ can be neglected and, for unstirred components (short-orbits due to e.g. line-of-sight), the terms with $\langle Z_{21}^{(j)} \rangle$ (small but non-null) survive [?]. Weak fluctuations yield

$$Z_{c,2} \approx \frac{\langle Z^{(2)} \rangle}{\langle Z^{(2)} \rangle + \langle Z_{22}^{(1)} \rangle} \left(\delta Z_{21,r}^{(1)} + j \delta Z_{21,i}^{(1)} \right) , \tag{18}$$

since in the denominator $\delta Z^{(2)}$ and $\delta Z_{22}^{(1)}$ are small if compared with their associated mean values, and where $j = \sqrt{-1}$.

The statistical expression (18) gives a clear picture of the fluctuation structure featuring the coupling impedance (and then, the coupled field): *its distribution function* (DF) *can be approximated by the renormalized* DF *of the forward cross-impedance characterizing the first cavity*, viz.

$$f_{Z_{c,2}}(z_{c,2}) \approx f_{Z_{c,2}}\left(\mathcal{A} z_{21}^{(1)} \right) , \tag{19}$$

with $f(\cdot)$ probability density function (pdf) of the random variables z, and

$$\mathcal{A} = \frac{\langle Z^{(2)} \rangle}{\langle Z^{(2)} \rangle + \langle Z_{22}^{(1)} \rangle} . \tag{20}$$

C. Three interconnected elements

The coupling impedance of two interconnected cavities is useful to characterize the *pure* fluctuation of the coupled field. In a real scenario, the transmission between two complex environments can take place through single- or multi-mode interconnections (waveguides, cables), as well as passing through a random overmoded environments.

Adopting the three-cavities model allows for characterizing practical scenarios. Let us consider the situation of three connected cavities, depicted in Fig. 2. On application of constitutive equations and boundary conditions at the two (connecting) sections yields

$$V_1^{(3)} = \frac{Z^{(3)} Z_{21}^{(2)} Z_{21}^{(1)}}{\left(Z_{22}^{(1)} + Z_{11}^{(2)} \right) \left(Z_{22}^{(2)} + Z^{(3)} \right) - Z_{12}^{(2)} Z_{21}^{(2)}} I_1^{(1)} . \tag{21}$$

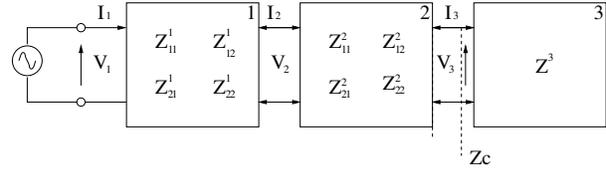The denominator of (21) exhibits the determinant of the

Fig. 2. Network model of three *coupled* chaotic elements.

following matrix relating the leakage currents, viz.,

$$\begin{bmatrix} \left(Z_{22}^{(1)} + Z_{11}^{(2)} \right) & -Z_{12}^{(2)} \\ -Z_{21}^{(2)} & \left(Z_{22}^{(2)} + Z^{(3)} \right) \end{bmatrix} \begin{bmatrix} I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} -Z_{21}^{(1)} I_1 \\ 0 \end{bmatrix} , \tag{22}$$

where a *current coupling matrix* $\underline{\underline{M}}_2$ arises

$$\underline{\underline{M}}_2 I = \underline{V} . \tag{23}$$

Thus, we obtain the following compact expression for the coupling voltage

$$V_3 = \frac{Z^{(3)} Z_{21}^{(2)} Z_{21}^{(1)}}{\Delta \underline{\underline{M}}_2} I_1 . \tag{24}$$

Again, application of small fluctuation theory is pertinent to the presence of losses (further enhanced by the coupling leakage itself), yielding

$$Z_{c,3} \approx \frac{\langle Z^{(3)} \rangle \delta Z_{21}^{(2)} \delta Z_{21}^{(1)}}{\langle M_{11} \rangle \langle M_{22} \rangle} , \tag{25}$$

where average quantities $<>$ involve only radiation impedances, and short orbits [?]. From (25), we argue that (the real or imaginary part of) coupling impedance fluctuates, in the presence of high losses, as the renormalized product of Gaussian DFs, i.e., following a Meijer-G function law. Such an approximation is also supported by the fact that the coupling takes place forwardly along the chain because of losses (small backward signal) [11], thus $\left| \delta Z_{12}^{(2)} \right| \approx 0$. In [6], it has been shown that cavity impedances are related to the *non-universal* features of each single port multiplied by a *universal fluctuating* term, viz.

$$Z_{ij}^{(c)} = \sqrt{R_{ii}^{(c)}} \, \xi^{(c)} \sqrt{R_{jj}^{(c)}} , \tag{26}$$

$$Z_{ii}^{(c)} = j X_{ii}^{(c)} + R_{ii}^{(c)} \xi^{(c)} , \tag{27}$$

where the impedance parameter of cavity c relates the port i with the port j, $R_{ii}^{(c)}$ is the real part of $Z_{ii}^{(c)}$, $R_{jj}^{(c)}$ is the real part of $Z_{jj}^{(c)}$, and ξ is a random variable. Note that the fluctuations in cavity c, $\xi^{(c)}$, depend only on the loss parameters of that cavities [6], [7], [12], [13].

Of course, in the weak fluctuation limit, the fact that $\langle Z_{21}^{(1)} \rangle = \langle Z_{12}^{(2)} \rangle = \langle Z_{21}^{(2)} \rangle = 0$, and the forward coupling, leads to the pdf of Z_c

$$f(z_{c,3}) \approx f\left(\frac{\langle Z^{(3)} \rangle}{\langle M_{11} \rangle \langle M_{22} \rangle} z_{21}^{(2)} z_{21}^{(1)} \right) , \tag{28}$$

with $\langle M_{11}\rangle$ and $\langle M_{22}\rangle$ are diagonal entries of the current coupling matrix, provided the weak forward coupling condition $\left|\delta Z_{21}^{(2)}\delta Z_{12}^{(2)}\right| << \langle M_{11}\rangle\langle M_{22}\rangle$ is satisfied. With these remarks, and applying the small fluctuations theory, pdf (28) can be recast as

$$f_{Z_{c,3}}\left(z_{c,3}\right)\approx f\left(\frac{\langle Z^{(3)}\rangle\sqrt{R_{22}^{(2)}R_{11}^{(2)}}\sqrt{R_{22}^{(1)}R_{11}^{(1)}}}{\langle M_{11}\rangle\langle M_{22}\rangle}\,\xi^{(2)}\,\xi^{(1)}\right)$$
$$= f\left(\mathcal{N}\,\xi^{(2)}\,\xi^{(1)}\right),\tag{29}$$

where the renormalization factor emphasizes separate contributions by each single port in each single cavity. The universal fluctuation $\xi^{(i)}$ is Gaussian distributed in high-losses case, while it is Lorentz-Cauchy distributed in lossless case [6]. Concerning the three-cavities chain, a very good agreement has been found between numerical simulation and the following law

$$f_{Z_{c,3}}\left(\mathrm{Im}\,[x]\right)\approx\frac{K_0\left(\frac{\mathrm{Im}[x]}{\sigma_{Z_{21}^1}\sigma_{Z_{21}^2}}\right)}{\pi\sigma_{Z_{21}^1}\sigma_{Z_{21}^2}},\tag{30}$$

expressing the solution of the Meijer-G path integral for the product of two Gaussian distributions, where K_0 is the 0-th order Bessel function of the first kind, and $x=z_c/\mathcal{N}$. It is interesting to notice the presence of a logarithmic singularity at the origin.

The calculation of a bivariate pdf for the real and imaginary parts of the coupling impedance is also possible for the product of two *complex* random variables, upon a generalized Mellin transform [14]

$$f_{Z_c}\left(\mathrm{Re}[z_c],\mathrm{Im}[z_c]\right)\approx\frac{\mathcal{N}}{\left(1+\mathrm{Im}[z_c]^2+\mathrm{Re}[z_c]^2\right)^2}.\tag{31}$$

III. COUPLED POWER AND INFLUENCE OF THE LOSS FACTORS

We wish to find the connection between the coupled signal and the power entering each cavity, in the case of two connected cavities.

Taking the absolute value of the coupling impedance, after some algebra we get

$$f_{|Z_c|^2}\left(|z_c|^2\right)\approx f\left(|\mathcal{A}|^2\left|z_{21}^{(1)}\right|^2\right),\tag{32}$$

meaning that the distribution of the coupling impedance squared magnitude is a rescaled version of the cross-impedance squared magnitude distribution. Since real and imaginary part of Z_{21} are Gaussian distributed for the case of high-losses, upon variate transforming one obtains that $|Z_{21}|$ is Rayleigh distributed, yielding

$$f_{|Z_c|;\sigma}\left(|z_c|\right)=\frac{|z_c|}{\sigma}\,e^{-|z_c|^2/\sigma^2}\approx\frac{|\mathcal{A}|\,|z_{21}|}{\sigma}\,e^{-(|\mathcal{A}||z_{21}|)^2/\sigma^2},\tag{33}$$
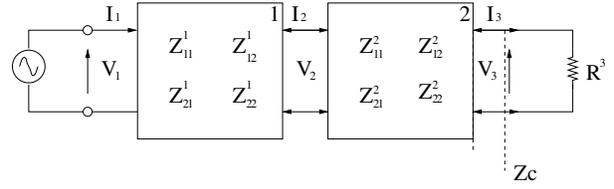

Fig. 3. Network model of three *coupled* chaotic elements.

where σ is related to the loss factor through the mean value and the standard deviation of $|Z_{21}|$ and $|\mathcal{A}|$ [6], [7]

$$\langle|Z_{21}|\rangle=\frac{\sigma}{|\mathcal{A}|}\sqrt{\frac{\pi}{2}}\tag{34}$$

$$\mathrm{Std}\left(|Z_{21}|\right)=\frac{\sigma}{|\mathcal{A}|}\sqrt{\frac{4-\pi}{2}},\tag{35}$$

while $|Z_c|^2$ (and then the coupled power) is exponentially distributed [15]

$$f_{|Z_c|^2;\sigma}\left(|z_c|^2\right)=\frac{1}{\beta}\,e^{-|z_c|^2/\beta}\approx\frac{1}{\beta}\,e^{-(|\mathcal{A}||z_{21}|)^2/\beta},\tag{36}$$

where, again, β is related to the quality factor through the mean value and the standard deviation of $|Z_{21}|$ and $|\mathcal{A}|$

$$\left\langle|Z_{21}|^2\right\rangle=\frac{\beta}{|\mathcal{A}|}\tag{37}$$

$$\mathrm{Std}\left(|Z_{21}|^2\right)=\frac{\beta^2}{|\mathcal{A}|^2}.\tag{38}$$

These results are consistent with the scalar Cartesian field and energy of a single [16], [3] and double [17] electromagnetic reverberation chamber. We also notice that σ and β can be easily derived in terms of radiation impedances and loss factors, e.g., from [6, Eq. (6.12)], the variance theorem, and (37) we have

$$\beta=|\mathcal{A}|\,\frac{R_{11}^{rad}R_{22}^{rad}}{\pi\alpha}.\tag{39}$$

with loss factor defined as $\alpha=\delta f_{\mathrm{3dB}}/\Delta f$, where $\delta f_{3-\mathrm{dB}}$ is a typical 3-dB bandwidth of a resonant mode, and Δf is the mean spacing between modes [6], [12].

IV. NESTED REVERBERATION CHAMBER WITH SMALL APERTURE

As a specific example of practical interest, we focus on the nested reverberation chamber [17]. We further assume that the coupling aperture is electrically small, so that the port approximation and the direct connection approximation ($\tau\approx 0$) apply. If the operation frequency is higher than the lowest usable frequency (LUF) of the smallest chamber [18], we obtain two weakly coupled overmoded chambers with different loss factor α: the outer one is excited by a small antenna, and the field in the inner one is picked up by a small antenna. The complex voltage measured at the antenna terminals (connected to a calibrated VNA port) is V_3 in Fig. 3, obtained by replacing the third element of the chain in Fig. 2 with the input impedance of the VNA. Furthermore, we

notice that measured voltage $V_3 = R^{(3)}I_3$ is equivalent to the coupling voltage $V_{c,3} = Z_{c,3}I_1$ related to the excitation current through the coupling impedance. Therefore, given the circuitry characteristics of the adopted instrument, the coupling signal properties can be derived upon calculation of $Z_{c,3}$. Thus, exact (24) and (25) approximate (high-lossy limit) expressions predict the coupled voltage fluctuations when the chambers get mode-stirred. Related DFs can be easily compared with measurements, given radiation impedances of antennae and aperture, and dividing over the deterministic excitation current. In the next section, we will present Montecarlo simulations of the coupling impedance fluctuation.

V. MONTECARLO SIMULATION

In this Section, we develop the simulation strategy for validating theoretical findings on coupling impedance distribution, and to verify the small fluctuation approximation employed in lossy chains modeling. In [12], a Montecarlo method has been presented and discussed. We basically use the same strategy for generating the single impedances of each cavity, and combine them according to the theoretical expressions obtained for two and three interconnected cavities. In general, the impedance matrix related to a M-ports cavity reads [7]

$$\underline{\underline{Z}} = \frac{-j}{\pi} \frac{\underline{\underline{W}}\,\underline{\underline{W}}^T}{\underline{\underline{\lambda}} - j\alpha\underline{\underline{1}}} \,, \tag{40}$$

where each entry of the matrix $\underline{\underline{W}}$ represents the coupling between the ith driving port and the jth eigenmode of the cavity, behaving as an independent Gaussian-distributed random variable with zero mean and unit variance, according to RMT assumptions [8]. The matrix $\underline{\underline{\lambda}}$ is a set of uniformly distributed eigenvalues – i.e., mapping of eigenvalues having semi-circle distribution [19].

In our simulations, we generated 600 random eigenvalues for each cavity of the chain, and build the associated random Green's function selecting a value for the loss factor α. We assumed identical cavities (homogeneous chain), with same loss factor, identical radiation resistance $R_{ii} = 50\ \Omega$, and $X_{ii} = 20\Omega$ for each port i.

Fig. 4 and 5 shows the pdf comparison for real and imaginary parts of the coupling impedance for two connected cavities and $\alpha = 10$. As predicted by the theory, this pdf follows the renormalized distribution of the complex cross-impedance, i.e., in the high-losses limit, a Gaussian DF [20]. This is confirmed by the adopted approximation theory. In the weak fluctuation limit, exact and approximate pdf of coupling impedance are found to be in excellent agreement even with a small number of samples. The coupling of chaotic cavities clearly highlights *a reduction in the magnitude of coupling fluctuation* compared to the uncoupled case, as a function of the number of elements in the chain. The comparison of $\text{Im}[Z_c]$ for three cavities is shown in Fig. 6. In the passage from 2 to 3 cavities, the distribution shape is dramatically changed: we notice squeezing and improved peakedness. This is clearly due to the product of cross-impedance fluctuations. Hence, it is worth making a direct comparison of the theoretical law with the Montecarlo
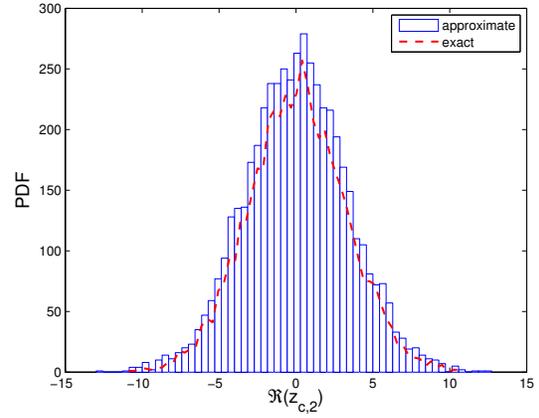

Fig. 4. Comparison between exact and approximate coupling impedance distribution DF in high-losses case (weak fluctuation limit): real part for two connected cavities for $\alpha = 10$ and 5000 samples. The random variable z_c has $[\Omega]$ units.
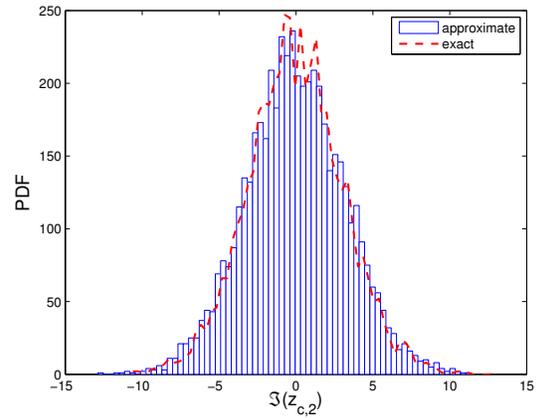

Fig. 5. Comparison between exact and approximate coupling impedance distribution in high-losses case (weak fluctuation limit): imaginary part for two connected cavities for $\alpha = 10$ and 5000 samples.
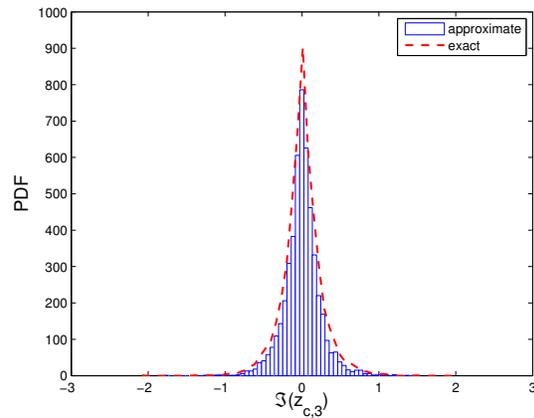

Fig. 6. Comparison between exact and approximate coupling impedance distribution in high-losses case (weak fluctuation limit): imaginary part for a chain of three cavities for $\alpha = 10$ and 5000 samples.
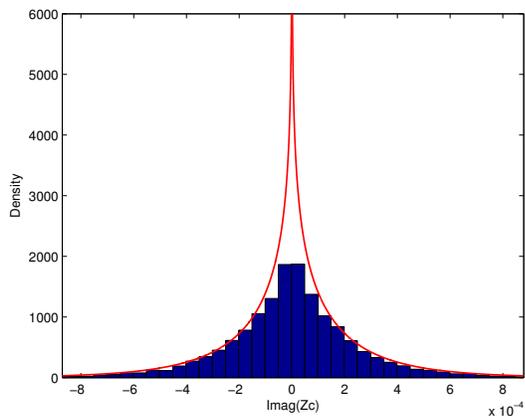
Fig. 7. Imaginary part of the coupling impedance: probability density function for a chain of three identical cavities with $\alpha = 100$, simulated generating 40000 samples. Theoretical DF fits very well with the numerical DF.
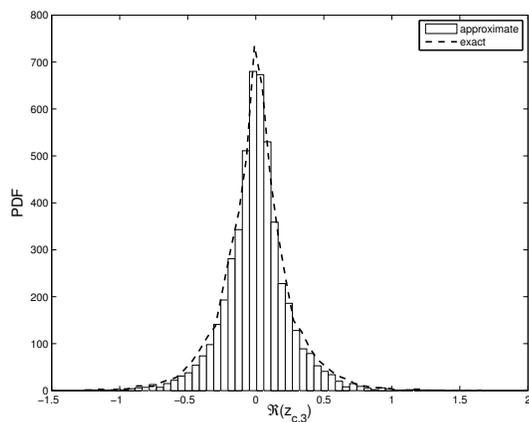

Fig. 8. Comparison between exact and approximate coupling impedance distribution in high-losses case (weak fluctuation limit): simulation of a VNA provision for cavities with $\alpha = 10$ and 5000 cavity realizations.

generated distribution. Fig. 7 shows a very good agreement between theoretical and numerical results, despite the lack of sampling near the origin, where a singular behavior occurs. A comparison with measurements would require accounting for radiation impedances, input impedance of the VNA, and multiplying by the excitation current. Therefore, the distributions presented here can be easily renormalized for predicting experimental fields measured in a weakly coupled reverberation chamber. Fig. 8 shows exact and approximate distributions of the real part of $Z_{c,3}$ ($V_3/I_1^{(1)}$) for $Z^{(3)} = R^{(3)} = 50\ \Omega$, mimicking the input impedance of a VNA port. In this case, the VNA would be connected to a receiving antenna inside the central cavity (inner reverberation chamber), connected to the first one (outer chamber) through a small aperture. The transmitting antenna is modeled by the first port, whose exciting current is deterministic as opposed to stochastic. The computation time is about 2 hours in a commercial laptop, as

the Montecarlo generation of cavity realizations is very fast.

VI. CONCLUSION

We derived closed-form expressions for the statistics of coupling impedance of two- and three-connected complex cavities. This has been carried out upon joint application of random matrix theory and electrical network theory. The coupling is assumed to take place through a cable or an electrically small aperture. In this scenario, also modeling weakly coupled reverberation chambers, the interconnection between cavities is direct ($\tau = 0$). Since the inner field of each cavity is governed by a random/chaotic scattering [21], the impedances involved in the calculation of the coupling impedance are random variables. In the high-loss limit, the weak fluctuation and the forward coupling approximation can be used to derive the coupled field statistics. Here, we derived the approximate probability density function of the coupling impedances: this is governed by the fluctuation of the (forward) cross-impedances of each single cavity in the chain. Results are of interest in evaluation of coupling among complex systems, nested reverberation chambers, and immunity/shielding test methods in highly overmoded environments[18].

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