Random coupling model for the radiation of irregular apertures

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***Abstract*—In this paper, we propose and investigate the radiation of *chaotic* apertures. It is assumed that an aperture is wide and its geometry is irregular enough to infer a random behavior of the tangential field, that is expanded in *chaotic* modes. Particular emphasis is devoted to the calculation of the free-space aperture admittance matrix, whose element average takes a simple closed-form expression. The radiation admittance matrix is found to be purely diagonal at relatively short-wavelength, and it exhibits unusual frequency behavior. The extreme scenario of a chaotic aperture radiating inside a chaotic cavity is analyzed by the random coupling model. *Transmitted* power distribution is generated in various loss conditions, upon oblique plane-wave excitation of the aperture. It is found that cavity loss and number of aperture modes influence symmetry and fluctuation law of the transmitted power distribution. Obtained results offer a mathematical framework for the physical understanding of scattering in extremely complicated environments, mode-stirred reverberation chambers, wireless channels, radar traces, and statistical optics.**

I. Introduction

Irradiation of apertures in free- and confined-space constitutes a fundamental mechanism of coupling between propagation environments in many physical scenarios. Aperture shape, geometry, and electrical dimensions are of special interest in, and strongly affect, the field propagation in several real-life scenario.

In this study, we present a theoretical model of apertures illuminated by relatively short-wavelength radiation. It is reasonable to think of practical situations where shape and location of openings are not known in great detail, posing difficulties to their analysis. Here, we can distinguish between two scenarios: one where the aperture shape is relatively simple but not completely regular, for which a mathematical expression of the inner fields is not available, one where the aperture boundary is so structured that its geometry cannot be reproduced exactly. The first scenario can be easily investigated by numerical methods such as the finite-difference time-domain (FDTD), and the finite elements method (FEM). The second scenario would require large effort to be tackled numerically. In either scenario, simple analytical expressions of equivalent network parameters would be beneficial [1]. Interestingly, beside analysis, also the synthesis of apertures can take advantage of irregular shapes. The ones we have in mind for this study are inspired by microwave billiards (2D) used to explore quantum and wave chaos of complex physical systems, e.g., Bunimovich, Sinai, and cardioid billiards [2]. In this work, we develop a statistical model for calculating the elements of the free-space radiation admittance of irregular apertures. We further propose an easy way to study the irradiation of such apertures inside irregular cavities. In particular, we use the random coupling model for calculating the cavity admittance matrix in a compact form, then we derive an expression of the power transmitted inside the cavity in terms of both radiation and cavity admittance matrices. The random coupling model, where the exact cavity mode spectrum is also replaced by a “chaotic” spectrum of the Gaussian Orthogonal Ensemble (GOE) [2], has proven to be an effective tool for predicting circuit interferences within a complicated enclosure [3]. For irregular apertures, the cavity admittance becomes a random variable through both aperture and cavity fields. We then average over aperture realization to focus on the influence of aperture irregularity to the power entering the cavity. It is shown how the number of chaotic aperture modes influence the fluctuation law of the power transmitted inside the cavity. The generation of power samples by our method is extremely fast and dependent on a few parameters of the systems. The simple formulas we derive can also be used to test numerical codes in very complicated geometries where meshing and discrete electromagnetic (full-wave) solutions are not trivial.

II. Free-space (radiation) admittance matrix

Consider an electrically large aperture of arbitrary irregular geometry, illuminated by external radiation at wavelength $\mathbf{k}^{inc}$. Assume the canonical situation where this external radiation is an oblique plane-wave $\mathbf{H}^{inc} = \mathbf{h}^{inc} \exp\left(i\mathbf{k}^{inc} \cdot \mathbf{x}_\perp\right)$, with $\mathbf{x}_\perp$ indicating the in-plane (aperture) coordinates. This scenario is illustrated in the scheme of Fig. 1, where the metallic plane hosting the aperture is treated as infinitely extended. The electromagnetic field induced by the external radiation in the aperture, transverse to $\hat{z}$, can be expanded in a basis of modes

$$\mathbf{E}_t\left(\mathbf{x}_\perp\right) = \sum_s V_s \mathbf{e}_s\left(\mathbf{x}_\perp\right) \ , \tag{1}$$

$$\mathbf{H}_t\left(\mathbf{x}_\perp\right) = \sum_s I_s \hat{z} \times \mathbf{e}_s\left(\mathbf{x}_\perp\right) \ , \tag{2}$$

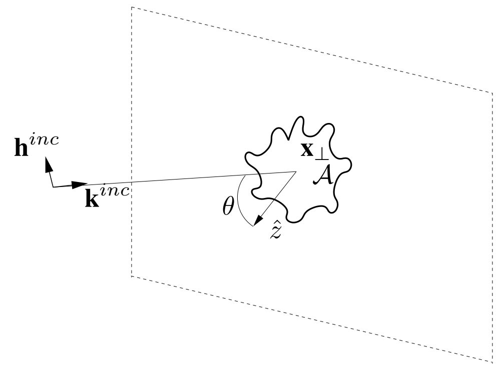

Fig. 1. Geometry of an irregular aperture illuminated by an external plane-wave.

where V_s and I_s represent the electric and magnetic mode amplitudes respectively. By solving the Maxwell equations in free-space with boundary conditions (1), we can evaluate the magnetic field on the plane $z = 0$ to find the magnetic mode amplitude. By taking the Fourier transform, this results in a matrix relation between I_s and V_s in k-space

$$I_s = \sum_{s'} Y^{rad}_{ss'}(k_0)\, V_s \ , \tag{3}$$

where $k_0 = \left|\mathbf{k}^{inc}\right|$, and $Y^{rad}_{ss'}$ is an element of the radiation admittance matrix. In particular, it can be proven that the radiation conductance takes the form [4], [5]

$$G^{rad}_{ss'}(k_0) = \Re\{Y^{rad}_{ss'}(k_0)\} = \sqrt{\frac{\epsilon}{\mu}} \int \frac{k_0^2 d\Omega_k}{8\pi^2}\ \tilde{\mathbf{e}}_s \cdot \underline{\underline{\Delta}}_{G^{rad}} \cdot \tilde{\mathbf{e}}^*_{s'} \ , \tag{4}$$

where there appears the Fourier transform of the mode basis function $\tilde{\mathbf{e}}_s(k_0)$, and the dyadic operator $\underline{\underline{\Delta}}_{G^{rad}} = \left[(\mathbf{k}_\perp \mathbf{k}_\perp)/k_\perp^2\right] + \left[(\mathbf{k} \times \hat{n})(\mathbf{k} \times \hat{n})/k_\perp^2\right]$. Similarly, by taking the Cauchy principal value of $Y^{rad}_{ss'}$ reveals that the radiation susceptance $B^{rad}_{ss'} = \Im\{Y^{rad}_{ss'}(k_0)\}$ formally represents the *Hilbert* transform of the radiation conductance. It is thus clear that the choice of the aperture basis function constitutes a critical point for the calculation of $G^{rad}_{ss'}$, and the subsequent calculation of $B^{rad}_{ss'}$.

In previous work, we calculated the elements of $\underline{\underline{Y}}^{rad}$, and investigated its structure for narrow elongated apertures [5]. Here, we are concerned with the derivation of a closed-form expression of $Y^{rad}_{ss'}$ when the aperture gets irregular and wide in all its dimensions. We also want to make those results general and independent on aperture geometry and details. Our main assumption in specifying a *chaotic* basis of modes for our problem is that *the aperture boundary is such irregular that* $\mathbf{e}_s$ *can be thought as made of a superposition of random plane waves, i.e., rays propagating along chaotic trajectories.* This is the so-called Berry's hypothesis, widely used in systems whose classical dynamics experiences exponential divergence [6]. Referring to Fig. 1, we can thus expand $\mathbf{e}_s$ into transverse electric (TE) and transverse magnetic (TM) plane-wave superpositions, viz.,

$$\mathbf{e}_s^{(TE,TM)}(\mathbf{x}_\perp) = \lim_{N\to\infty} \frac{\beta}{\sqrt{N}} \sum_{n=1}^{N} \begin{bmatrix} (\hat{z} \times \mathbf{k}_n)^{(TE)} \\ (\mathbf{k}_n)^{(TM)} \end{bmatrix} \times \zeta_n \cos(\mathbf{k}_n \cdot \mathbf{x}_\perp + \theta_n) \ , \tag{5}$$

where ζ_n, $\mathbf{k}_n$, and θ_n are uncorrelated random variables. The normalization constant is found to be $\beta = \sqrt{N} = \sqrt{2/(k_s^2 \mathcal{A})}$.

Then, following the procedure drawn in [5, Sec. II.B], we calculate the Fourier transform of (5). Instead of doing that for $\mathbf{e}_s = \mathbf{e}_s^{(TE)} + \mathbf{e}_s^{(TM)}$, we better calculate the Fourier transform of the correlation tensor

$$\underline{\underline{C}}_{ss'} = \left\langle \mathbf{e}_s(x_\perp)\, \mathbf{e}_s\left(x'_\perp\right) \right\rangle_{\mathcal{A}} = \delta_{ss'} \frac{\underline{\underline{I}}}{2\mathcal{A}} J_0\left(k_s \left|\mathbf{x}_\perp - \mathbf{x}'_\perp\right|\right) \ , \tag{6}$$

where $\langle\cdot\rangle_{\mathcal{A}}$ stands for the ensemble average over statistically equivalent aperture realizations, performed by assuming uncorrelated pairs of random plane-waves [2, Sec. 6]

$$\tilde{\underline{\underline{C}}}_{ss'}(k_0) = \delta_{ss'}\, \underline{\underline{I}}\, \frac{\pi}{k_s}\, \delta(k_0 - k_s) \ , \tag{7}$$

where k_s is the s-th resonance wavenumber of the aperture. The spatial correlation of (6) is consistent with the well-known results for microwave billiards [2, Sec. 6].

We note that the random the plane-wave spectrum results in a *purely diagonal* radiation admittance matrix, that strongly simplifies the dyadic kernel of (4) after taking the aperture ensemble average

$$\left\langle G^{rad}_{ss'}(k_0) \right\rangle_{\mathcal{A}} = \sqrt{\frac{\epsilon}{\mu}} \int \frac{k_0^2 d\Omega_k}{8\pi^2}\ \tilde{C}_{ss'}\, \underline{\underline{I}} : \underline{\underline{\Delta}}_{G^{rad}} \ , \tag{8}$$

where the double dot dyadic product reads

$$\underline{\underline{I}} : \underline{\underline{\Delta}}_{G^{rad}} = 1 + \frac{k_0^2 - k_\perp^2}{k_0^2} \ , \tag{9}$$

Inserting (7) and (9) into (8) yields, upon the sampling property of Dirac's delta

$$\left\langle G^{rad}_{ss'}(k_0) \right\rangle_{\mathcal{A}} = \frac{1}{2}\, \delta_{ss'} \sqrt{\frac{\epsilon}{\mu}} \frac{k_0}{\sqrt{k_0^2 - k_s^2}} \left(2 - \frac{k_s^2}{k_0^2}\right) \ , \tag{10}$$

whose frequency behavior is represented in Fig. 2 for the normalized wavenumber $k_s = 1$. We see that, in the very high-frequency limit $k_0 \gg k_s$, the free-space conductance $\left\langle G^{rad}_{ss'}(\infty) \right\rangle_{\mathcal{A}} = \sqrt{\epsilon/\mu} = G_0 \approx 2.7\ \mathrm{m}\Omega^{-1}$ is perfectly retrieved asymptotically. It should be pointed out that (10) does not exhibit explicit dependence on the aperture dimensions. We will discuss how to reintroduce this dependence in our statistical formulation through the distribution of k_s. The susceptance $\left\langle B^{rad}_{ss'}(k_0) \right\rangle_{\mathcal{A}}$ is computed numerically by Hilbert transform as in [5].

III. Radiation inside a cavity: Random Coupling Model

We now solve the boundary-value problem shown in Fig. 3, where the cavity we consider is three-dimensional and, similarly to the aperture, its boundaries are irregular enough to create a wave chaotic regime.

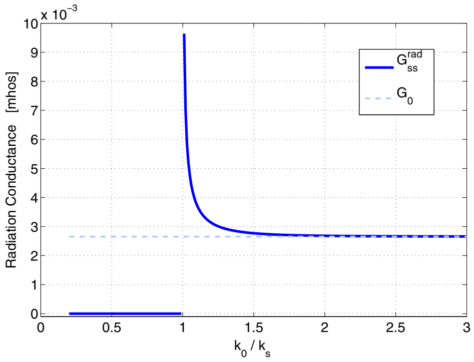

Fig. 2. Frequency behavior of the radiation conductance for irregular apertures.

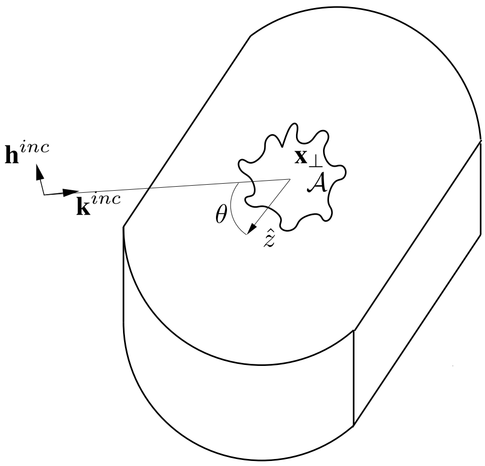

Fig. 3. Geometry of a 2D irregular aperture illuminated by an external plane-wave, radiating inside a 3D (Bunimovich-like) chaotic cavity.

By applying the Berry conjecture to replace an exact cavity mode with a chaotic mode, and RMT to replace the actual cavity spectrum with that of a chaotic system (typically GOE, GUE), it has been proven that the aperture admittance matrix for three-dimensional cavities splits as [5]

$$\underline{\underline{Y}}^{cav,\alpha} = i\underline{\underline{B}}^{rad} + \left[\underline{\underline{G}}^{rad}\right]^{1/2} \cdot \underline{\underline{\xi}} \cdot \left[\underline{\underline{G}}^{rad}\right]^{1/2} \quad . \tag{11}$$

In (11) the chaotic scattering is modeled by the fluctuating $N_{\mathcal{A}} \times N_{\mathcal{A}}$ matrix, defined as a *chaotic* modes superposition [7]

$$\underline{\underline{\xi}} = -\frac{i}{\pi} \sum_n \frac{\underline{\Phi}_n \underline{\Phi}_n^T}{\mathcal{K}_0^2 - \mathcal{K}_n^2 + i\alpha} \ , \tag{12}$$

where $\underline{\Phi}_n$ is a $N_{\mathcal{A}} \times M$ matrix expressing the interactions of the $N_{\mathcal{A}}$ aperture modes with the M chaotic cavity eigenmodes, $\mathcal{K}_{(\cdot)}^2 = k_{(\cdot)}^2/\Delta k^2$, and the loss factor is defined as the ratio of the average cavity quality factor width over the average cavity mean spacing between nearest-neighbour modes Δk^2, viz.,

$$\alpha = \frac{k_0^2}{Q \Delta k^2} \ . \tag{13}$$

The "universal" matrix (12) fluctuates over the cavity ensemble [8]. Such behavior can be achieved in practice, for example, by mode-stirring a regular enclosure [9]. It is now clear that the moments of $\underline{\underline{Y}}^{cav,\alpha}$ can be calculated by ensemble averaging over both aperture and cavity realizations. Concerning the first-order central moment, such a *superstatistics* would still lead the radiation case

$$\left\langle \left\langle \underline{\underline{Y}}^{cav,\alpha} \right\rangle_{\mathcal{A}} \right\rangle_{\mathcal{V}} = \left\langle \underline{\underline{Y}}^{rad} \right\rangle_{\mathcal{A}} \ , \tag{14}$$

a remarkable property of RCM [7], also valid when $\alpha \to \infty$.

The difference with our previous works on aperture is that, now, since the aperture is irregular, its spectrum of natural modes is replaced by a spectrum of chaotic modes. Therefore, $\underline{\underline{Y}}^{cav}$ becomes a random matrix containing fluctuations in both the *aperture* and *cavity* fields.

As for regular geometries, also the cross-section of irregular apertures should be equal to the geometrical area at high-frequencies [10]. In presence of boundary irregularity, the influence of the number of modes on the cavity field fluctuations in the presence of losses is also of fundamental interest. An example is given in the next section, where we calculate the power transmitted by an irregular aperture when it radiates inside a complex cavity. Furthermore, our findings could be useful to tackle problems involving *overmoded* regular apertures, for which the basis of modes in (5) is valid.

IV. Power transmitted through the aperture

Consider external radiation exciting the aperture depicted in Fig. 1. Typically, according to the Lamb cosine law, the power captured by a large aperture is proportional to $\mathcal{A}\cos(\theta)$. The external radiation induces a virtual magnetic current $\underline{I}^{inc}$ on the aperture plane. This current radiates both to the left- and to the right-space of the aperture plane located at $z = 0$. In our problem, the right half-space could be open or confined by an irregular cavity, while the left half-space, from where the radiation is incident, is always open. By imposing continuity of tangential fields across the aperture, projecting onto the aperture basis of modes, and Fourier transforming, the equivalent current takes the form $I_s^{inc} = -\hat{z} \cdot \tilde{\mathbf{e}}_s \left(-\mathbf{k}_\perp^{inc}\right) \times \mathbf{h}^{inc}$. The *transmitted* power is given by

$$P_t^\alpha = 4\, \underline{I}^{inc} \cdot \underline{\underline{Z}}^\alpha \cdot \underline{I}^{inc,T,*} \ , \tag{15}$$

whose dependence on the cavity loss is now made clear by the RCM, which gives

$$\underline{\underline{Z}}^\alpha = \left(\underline{\underline{Y}}^{rad} + \underline{\underline{Y}}^{cav,\alpha}\right)^{-1} \cdot \underline{\underline{Y}}^{cav,\alpha,*} \cdot \left(\underline{\underline{Y}}^{rad} + \underline{\underline{Y}}^{cav,\alpha}\right)^{-1,*} \ , \tag{16}$$

Once again, the power in (15) is a random variable of both aperture and cavity realizations. Assuming now that the elements of $\underline{\underline{Z}}^\alpha$, the equivalent impedance of the cavity-backed aperture, are statistically independent from the induced magnetic current, the aperture ensemble average yields

$$\langle P_t^\alpha \rangle_{\mathcal{A}} = 4 \sum_s \left\langle \mathcal{Z}^\alpha \left(k_s, \left|\mathbf{k}_\perp^{inc}\right|\right) \right\rangle_{\mathcal{A}} \left\langle \left| I^{inc} \left(k_s, \left|\mathbf{k}_\perp^{inc}\right|\right) \right|^2 \right\rangle_{\mathcal{A}} \ , \tag{17}$$

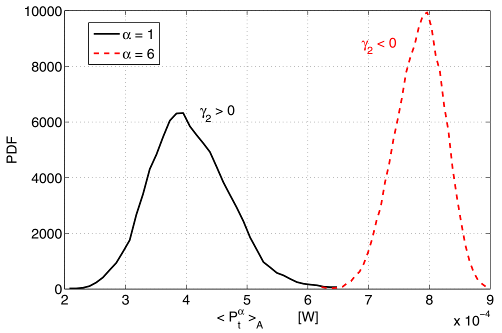

Fig. 4. Transmitted power distribution for $\mathcal{A} = 0.25$ m $\times 0.25$ m, $N_{\mathcal{A}} = 25$ modes, and $\theta = \pi/4$. The distribution kurtosis γ_2 changes its sign passing from low- ($\alpha \leq 1$) to high-loss ($\alpha \geq 3$) condition of the irregular cavity.

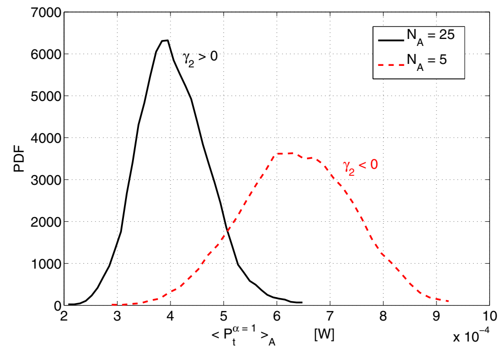

Fig. 5. Transmitted power distribution for $\mathcal{A} = 0.25$ m $\times 0.25$ m, $\alpha = 1.0$ modes, and $\theta = \pi/4$. The distribution kurtosis γ_2 changes its sign varying the number of aperture chaotic modes.

the second order moment of $\underline{I}^{inc}$ we evaluate using (7)

$$\left\langle I^{inc}\left(k_s, \left|\mathbf{k}_\perp^{inc}\right|\right) I^{inc,*}\left(k_{s'}, \left|\mathbf{k}_\perp^{inc}\right|\right)\right\rangle_{\mathcal{A}} \tag{18}$$

$$= \left|\mathbf{h}^{inc}\right|^2 \frac{2\pi}{k_s} \delta\left(k_s - k_{s'}\right) \delta\left(\left|\mathbf{k}_\perp^{inc}\right| - k_s\right) \ . \tag{19}$$

Finally, by converting the sum on aperture modes into an integral, and by using the sampling property of the Dirac delta, yields

$$\langle P_t^\alpha \rangle_{\mathcal{A}} = \frac{16\pi \left|\mathbf{h}^{inc}\right|^2}{\left|\mathbf{k}_\perp^{inc}\right|^2} \mathcal{D}\left(\left|\mathbf{k}_\perp^{inc}\right|\right) \mathcal{Z}^\alpha\left(\left|\mathbf{k}_\perp^{inc}\right|\right) \ , \tag{20}$$

where the aperture area enters through the mode density, whose expression is given by the smooth Weyl's law for a two-dimensional billiard

$$\mathcal{D}(k) = \frac{\mathcal{A}}{4\pi} k^2 \ . \tag{21}$$

Fig. 4 shows the power distribution for a cavity-backed irregular aperture of area $\mathcal{A} = 0.25$ m $\times 0.25$ m in two loss conditions, $\alpha = 1$ and $\alpha = 6$ at oblique incidence $\theta = \pi/4$. The incident power is $P_i = 10$ mW. By performing Monte Carlo computations of $\mathcal{Z}^\alpha$ in (20) it is found that going from low loss to high loss condition results in a change of the kurtosis sign of the power distribution, provided a relatively high (moderate) number of "chaotic" modes are excited in the aperture. Fig. 5 shows that also increasing the number of modes, i.e., increasing the frequency, results in a change on the kurtosis sign. Furthermore, the very sudden switching of "conductance channels" brought by the chaotic behavior strongly reduces the probability of detecting structure resonances (typically related to the aperture geometry). This situation is different from regular aperture radiating inside complicate but integrable cavities, where radiation resonances are hard to damp, even by using materials with losses.

V. Conclusion

We derived a closed-form expression of the radiation admittance for wide and irregular apertures by using the wave chaos theory. It turns out that the admittance matrix is purely diagonal and the magnetostatic susceptance is negligible, while the radiation susceptance is not. The extreme scenario of irregular apertures radiating inside wave chaotic enclosures has been investigated using the RCM. The cavity fluctuations are separated from the aperture fluctuations. Statistical independence between those two fluctuations leads to a closed-form expression of the power entering the cavity. We then generated power samples in various loss conditions and aperture excitations, resulting in a different number of modes excited in the aperture. We found that the power distribution kurtosis is highly affected by the cavity loss factor for a moderate number of aperture modes, and, in low-loss conditions, by the number of aperture chaotic modes. The derived formulas are simple and they can be used as a comparison term for numerical simulations of apertures whose shape is extremely complicated and would require a very refined and inhomogeneous mesh to be simulated at relatively high frequencies.

Acknowledgement

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