Microwave Basics

Microwave energy is transported by means of transmission lines.

**TEM**  No longitudinal component of $E$ or $B$
Requires two conductors

**TE, TM**  Present in waveguide (single conductor)

Coaxial Cable are the most common form of transmission line

The radial electric field is given by
\[
E = \frac{V}{\ln(b/a)} \frac{1}{r} \quad (V/m)
\]

The axial magnetic field is given by
\[
H = \frac{I}{2\pi} \frac{1}{r} \quad (A/m)
\]

*Fig. 1.1-3 (a)* is a nice picture of the coaxial TEM mode.
The speed of light in the coaxial cable is

\[ V_{ph} = \frac{c}{\sqrt{\varepsilon_r}} = \frac{1}{\sqrt{\varepsilon_0 \varepsilon_r}} \]

\( \varepsilon_r = \text{relative permittivity} \)

\( \mu_r = \text{relative permeability} \)

\[ \lambda f = V_{ph} \]

\[ \lambda_{TEM} = \frac{\lambda_{vac}}{\sqrt{\varepsilon_r}} \]

\[ V(t) = |V| e^{i(2\pi ft + \phi)} \]

\[ V(x,t) = |V| e^{i(2\pi ft + \phi - \beta x)} \]

\[ \beta = \frac{2\pi}{\lambda} \]

If a wave propagates a distance \( L \), its phase advances by \( \beta L = 2\pi L / \lambda \).

1.3 Characteristic Impedance

A property of a transmission line is its characteristic impedance. It is defined as the ratio of voltage to current at a given point:

\[ Z_c = \left. \frac{V}{I} \right|_{\text{point}} \]

True for a travelling wave. The characteristic
impedance for a standing wave is not constant, since that is a combination of two waves travelling in opposite directions. Their

\[ Z_c(x) = \frac{V_{total}(x)}{I_{total}(x)} \]

varies along the line.

For travelling waves,

Derive the equations for a travelling wave on a transmission line:

\[ L = \text{inductance/length} \]
\[ C = \text{capacitance/length} \]

Inductance is associated with the flux produced by the oppositely directed currents in the pair of conductors. When the currents vary in time, there is a voltage change along the line.

The change in voltage in the inductance times the time rate of change of the current.

\[ \text{Voltage change} = \frac{\Delta V}{\Delta z} \Delta z = -\left( L \frac{\Delta z}{\Delta t} \right) \frac{\Delta I}{\Delta t} \]
A change in current along the line is merely the current that is shunted across the distributed capacitance. The rate of decrease of the current with distance is the capacitance times the time rate of change of the voltage.

\[ \text{Current change} = \frac{dI}{dt} \Delta z = -(\varepsilon_0 \Delta z) \frac{dV}{dt} \]

So we have the two equations:

\[ \frac{dV}{dz} = -L \frac{dI}{dt} \]
\[ \frac{dI}{dz} = -C \frac{dV}{dt} \]

By taking further partial derivatives, one can get a wave equation for either \( V \) or \( I \):

\[ \frac{\partial^2 V}{\partial z^2} = \frac{1}{LC} \frac{\partial^2 V}{\partial t^2} = \frac{1}{V_{ph}^2} \frac{\partial^2 V}{\partial t^2} \]

\[ V_{ph} = \frac{1}{\sqrt{LC}} \]

Also:

\[ \frac{\partial^2 I}{\partial z^2} = \frac{1}{V_{ph}^2} \frac{\partial^2 I}{\partial t^2} \]

There are solutions of the form:

\[ V(z,t) \sim A e^{\frac{z-2z_{ph}}{V_{ph}}} + B e^{\frac{z+2z_{ph}}{V_{ph}}} \]
To find the current one uses one of the above equations:
\[
\frac{\partial V}{\partial t} = -L \frac{\partial i}{\partial t}
\]

\[
\frac{i}{V_{ph}} \left( -A e^{i(t - 2\pi/f_0)} + \beta e^{i(t + 2\pi/f_0)} \right) = -L \frac{\partial i}{\partial t}
\]

\[
I(2, t) = \frac{1}{LV_{ph}} \left( -A e^{i(t - 2\pi/f_0)} + \beta e^{i(t + 2\pi/f_0)} \right)
\]

So,
\[
I \sim \frac{V}{LV_{ph}} = \frac{V}{z_0}
\]

\[
z_0 = LV_{ph} = \sqrt{\frac{L}{C}}
\]

The characteristic impedance.

For a coaxial line

\[
C = \frac{q}{\Phi_{0, A}}
\]

\[
\Phi = -\int_{a}^{b} E_r dr = -\int_{c}^{b} \frac{q}{2\pi \varepsilon r} = \frac{q}{2\pi \varepsilon} \ln(b/a)
\]

\[
C = \frac{2\pi \varepsilon}{\ln(b/a)}
\]

\[
L = \frac{1}{\mu} \int_{S} \mathbf{B} \cdot d\mathbf{a}
\]

\[
L = \frac{1}{\mu} \int_{S} \frac{\mu}{2\pi r} dr dl = \frac{\mu}{2\pi} \int_{0}^{b} \frac{dr}{r} = \frac{\mu}{2\pi} \ln(b/a)
\]

\[
H_A = \frac{I}{2\pi r}
\]
So far a coxial cable;

\[ V_{ph} = \sqrt{\frac{1}{LC}} = \sqrt{\frac{\mu}{2\pi \varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}} \quad \text{same as free space!} \]

and

\[ Z_0 = \sqrt{\frac{\mu}{\varepsilon}} = \frac{\sqrt{\mu}}{2\pi \varepsilon \ln(\lambda/\lambda)} = \sqrt{\frac{\mu}{\varepsilon}} \cdot \frac{\ln(6/4)}{2\pi} \]

\[ M_0 = 4\pi \times 10^{-7} \text{ H/m} \]
\[ Z_0 = 8.85 \times 10^{-12} \text{ F/m} \]

\[ Z_{free} = 376.8 \Omega \]

Antenna must match this.

\[ Z_{0.055 \text{ coax}} = \sqrt{376.8 \Omega \cdot \frac{1}{\sqrt{2}} \cdot \frac{\ln(0.033/0.010)}{2\pi}} \]

\[ = 50.6 \Omega \]

Universal Standard

Losses and Attenuation:

Microwave signals vary in magnitude over many orders, so it is convenient to use logarithms to discuss their losses.

\[ \text{Loss (nepers)} = \frac{1}{2} \ln \frac{P_1}{P_2} \]

\[ \text{loss (dB)} = 8.686 \cdot \text{loss (nepers)} \]

\[ \text{loss (dB)} = 10 \log \frac{P_1}{P_2} \]
Suppose half of the power is lost in the transmission line.

\[ P_2 = P_1 / 2 \]

\[ \text{Loss (dB)} = 10 \log 2 = 3.01 \text{ dB} \]

3 dB in half power

Note that since \( P \propto E^2 \) or \( B^2 \)

\[ \text{Loss}_{E,B} (\text{dB}) = 20 \log \frac{E_2}{E_1} \]

The attenuation describes the loss in power per unit length. \( \Delta x \) (dB/length)

\[ \frac{P_2}{P_1} = 10^{-\frac{1}{10} \Delta x \times \text{Length}} \]

\[ \frac{V_2}{V_1} = \frac{E_2}{E_1} = 10^{-\frac{1}{20} \Delta x \times \text{Length}} \]

For a normal metal transmission line \( \Delta x \propto \sqrt{f} \)

For a superconducting transmission line \( \Delta x \propto f^2 \)
Dielectric Loss

\[ Z_e = \frac{1}{i\omega C} \]
\[ \sigma \sim i\omega C \]

\[ \bar{J} = \sigma \bar{E} \]
\[ \bar{J} = i\omega C \bar{E} \]

So the displacement current leads the voltage by 90° in an ideal lossless capacitor. If the dielectric is lossy, there is a real part of the conductivity, and there is a component of current in phase with the voltage:

\[ \bar{J} = (i\omega C + \sigma_{\text{real}}) \bar{E} \]

\[ C \sim \varepsilon_r \Rightarrow \varepsilon_r ' - i\varepsilon_r '' \quad \text{Complex Dielectric Constant} \]

\[ \tan \delta = \frac{\text{Loss Current}}{\text{Reactive Current}} \]

\[ \tan \delta = \frac{\varepsilon_r ''/\varepsilon_r '}{\varepsilon_r '} \]

\[ \varepsilon_r ' \quad \text{and} \quad \varepsilon_r '' \]

Attenuation in a dielectric is given by

\[ \alpha_{\text{die}} = \pi \nu_{th} f \tan \delta \quad (\text{neper/m}) \]

\[ \sim f^2 \]
Standing Waves

Before we were talking about travelling waves. But in reality, we have waves travelling in both directions on the transmission line \( \rightarrow \) standing waves.

Any discontinuity in the uniform transmission line gives rise to standing waves.

In microwave circuits, the concepts of voltage and current are meaningful only when a single mode is propagating. Must establish a reference plane. The ratio of the voltage to current at that plane is defined as the device's impedance.

\[
\begin{array}{c}
V^+ = 2c \\
I^+ = I_{term}
\end{array}
\]

For just one travelling wave going to the right.

\[
\frac{V_{term}}{I_{term}} = \frac{V^+}{I^+}
\]

at the terminating impedance

The only way for \( V^+ = V_{term} \) and \( I^+ = I_{term} \) is if \( 2c = \frac{V_{term}}{I_{term}} \), that is, the termination must have the same impedance as the coaxial cable.
If \( \frac{V}{I} \neq \frac{V}{I} \), then there must be a reflected wave. In that case:

\[
\frac{V^-}{I^-} = -Z_c
\]

The total voltage and current are then:

\[
V = V^+ + V^-
\]
\[
I = I^+ + I^-
\]

Require continuity at the terminating impedance:

\[
(V^+ + V^-)_{at \, Tem} = V_{Tem}
\]
\[
(I^+ + I^-)_{at \, Tem} = I_{Tem}
\]

Combine these equations:

\[
\frac{V^+}{I^+} + \frac{V^-}{I^-} = \frac{1}{Z_c} I_{Tem}
\]

\[
V^+ - \frac{V^-}{I^-} = 2Z_c I^+
\]
\[
V^- - \frac{V^+}{I^+} = 2Z_c I^-
\]

\[
\frac{I^+}{V^+} = \frac{1}{Z_c} \quad \frac{I^-}{V^-} = -\frac{1}{Z_c}
\]
\[
I^+ + I^- = \frac{1}{Z_c} (V^+ - V^-)
\]
\[
V^+ + V^- = \frac{2}{Z_c} \left( \frac{V^+ - V^-}{Z_c} \right)
\]
\[ V^+(1 - \frac{2T}{2c}) + V^-(1 + \frac{2T}{2c}) = 0 \]

\[
V^- = V^+\left(\frac{-1 + \frac{2T}{2c}}{1 + \frac{2T}{2c}}\right)
\]

\[ (\text{Ex}) \quad \text{Suppose } Z_{\text{term}} = Z_c, \text{ then } \]
\[ V^- = V^+\left(\frac{-1+1}{1+1}\right) = 0 \]

No reflection from a perfectly terminated line!

\[ (\text{Ex}) \quad \text{Suppose there is a short circuit at the load; i.e. } Z_{\text{term}} = 0 \]
\[ \sqrt{V_{\text{term}}} = 0 \]

Then
\[ V^- = V^+\left(\frac{-1}{1}\right) = -V^+ \]

and all of the signal is reflected, with a 180° phase shift!

\[ (\text{Ex}) \quad \text{Suppose the transmission line ends with an "open" i.e. } Z_{\text{term}} = \infty, \text{ then } \]
\[ V^- = V^+(1) \]

This produces a complete reflection with no phase shift.

\[ V^+_{\text{term}} = 0 \]
Ex) Suppose the transmission line is left open to free space, $Z_{tm} = 377 \Omega$

\[ V^- = V^+ \left( \frac{377/50 - 1}{377/50 + 1} \right) = 0.76 \ V^+ \]

\[ P^- = 0.586 \ P^+ \quad \Gamma = 0.76 \text{ reflection coeff.} \]

**Reflection Coefficient**: Ratio of the reflected to incident voltages

\[ \Gamma = \frac{V^-}{V^+} \]

so that

\[ \Gamma = \frac{2\pi/2e - 1}{2\pi/2e + 1} = |\Gamma| e^{i\theta} \]

\( \theta \) is the angle by which the reflected voltage leads the incident voltage.

\[ 0 \leq |\Gamma| \leq 1 \]

**Return Loss**

\[ R \] compares the power in the reflected wave to the power in the forward wave

\[ R(dB) = 10 \log_{10} \frac{\text{Incident Power}}{\text{Reflected Power}} = 10 \log_{10} \frac{|V^+|^2}{|V^-|^2} = 20 \log_{10} \frac{1}{\Gamma^2} \]

- Totally reflecting termination \( \rightarrow R = 0 \)
- Totally absorbing termination \( \rightarrow R \rightarrow \infty \)
Transforming voltage on a transmission line

Suppose we know the voltage and current at \( w_1 \), and we wish to calculate it at \( w_2 \).

The forward voltage changes in magnitude by \( e^{\alpha(w_2-w_1)} \)

The forward voltage changes in phase by \( \beta(w_2-w_1) \)

\[ V^+(w_2) = V^+(w_1) e^{\alpha(w_2-w_1)} e^{i\beta(w_2-w_1)} \]

\[ V^-(w_2) = V^-(w_1) e^{-\alpha(w_2-w_1)} e^{-i\beta(w_2-w_1)} \]

The reflection coefficient is given as

\[ \rho(w_2) = \rho(w_1) e^{-2\alpha(w_2-w_1)} e^{-2\beta(w_2-w_1)} \]

Transformer Rule:

\[ Z_{in} \quad Z_c \quad Z_{term} \]

\[ Z_{in} = \frac{Z_{term} + i\tan\beta l}{1 + iZ_{term} \tan\beta l} \]

\[ Z = Z/Z_c \]

The transmission line of length \( l \) transforms the impedance at the termination to \( Z_{in} \).
Examine the properties of various transmission lines.

Ex) Let $Z_{\text{ext}} = Z_c$, then $\overline{Z_{\text{in}}} = 1$ or $Z_{\text{in}} = Z_c$
for any length of line.
A perfectly terminated line has the impedance of the transmission line.

Ex) Suppose $\lambda = \lambda_0$ or $n\lambda_0$, then $\beta l = \pi$,
\[ \tan \beta l = 0 \] and
\[ \overline{Z_{\text{in}}} = Z_{\text{ext}} \]
so the transmission line is effectively "transparent."

Shaded and open transmission lines are practical and have just about any impedance you want
(for a given $\lambda$).