Wave Chaos Experiments with and without Time Reversal Symmetry: GUE and GOE Statistics

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It has been predicted that in the semiclassical regime the level statistics of a classically chaotic system correspond to that of the Gaussian unitary ensemble (GUE) of random matrices when time reversal symmetry is broken. This Letter presents the first experimental test of this prediction. The system employed is a microwave cavity containing a thin ferrite strip adjacent to one of the walls. When a sufficiently large magnetic field is applied to the ferrite (thus breaking the time reversal symmetry) good agreement with GUE statistics is obtained. The transition from Gaussian orthogonal ensemble (GOE) (which applies in the absence of the applied field) to GUE is also investigated.

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It has been conjectured that, for chaotic systems in the semiclassical limit, the spectral statistics of the Schrödinger equation correspond to that of random matrices with the same symmetry [1]. In particular, when the system is time reversible, the statistical fluctuations of the energy levels are conjectured to be the same as those for the "Gaussian orthogonal ensemble" (GOE) of random matrices. As a simple example of this class of systems, consider a charged particle in a scalar potential. By reversing the direction of the momentum of the particle, the classical particle will retrace its own path. The wave equation for this particle is real and the corresponding GOE consists of real random symmetric matrices. On the other hand, when a magnetic field $\mathbf{B}$ is applied, the time reversal symmetry is broken. A classical charged particle will no longer retrace its own path when the direction of its momentum is reversed. In this case, the Schrödinger equation is complex, $\mathbf{p} \rightarrow -i\hbar\nabla - q\mathbf{A}(\mathbf{r})$, and (in the absence of special symmetries) the statistical fluctuations of the energy levels are conjectured to be the same as those for the "Gaussian unitary ensemble" (GUE) of random Hermitian matrices.

Although the predictions of GOE statistics in actual physical systems have been observed by others [2–5], there has been no experimental verification of the GUE predictions. The purpose of the present work is to verify the GUE predictions in an experimental setting using a 2D microwave cavity with a thin magnetized ferrite strip adjacent to one of the walls. To see how a magnetized ferrite breaks the time reversal symmetry in the electromagnetic wave equation, consider the situation when a plane wave with the electric field $\mathbf{E} = E_z \exp(ik_x x + ik_y y)\hat{\mathbf{z}}$ perpendicular to the plane of incidence is incident from the left ($x < 0$) on a slab of magnetized ferrite ($0 < x < d$) which is placed adjacent to a perfect conductor on the right ($x = d$). In the presence of a static magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$ perpendicular to the plane of incidence, the magnetic permeability μ of the ferrite, in the absence of losses, is

$$\mu = \begin{bmatrix} \mu_{\|} & -i\kappa & 0 \\ i\kappa & \mu_{\|} & 0\ i \\ 0 & 0 & \mu_z \end{bmatrix}, \quad (1)$$

where $\mu_{\|}$, κ, and μ_z are real. At the interface between the ferrite and the empty cavity, the boundary conditions require the continuity of both E_z and the tangential component of $\mathbf{H}$, which, in the ferrite, is proportional to $(\mu_{\|}\partial E_z/\partial x + i\kappa\partial E_z/\partial y)$. One can then calculate the reflection coefficient $\Gamma = e^{i\phi(B)}$ of this plane wave. It can be shown that upon reversal of the direction of the incident wave (i.e., $k_y \rightarrow -k_y$), because of the nontrivial mixing of the partial derivatives of E_z at the boundary, the phase $\phi(B)$ is different. That is (unlike the situation with $B = 0$), when the direction of the ray is reversed, the phase shift changes, $\phi(B, k_y) \neq \phi(B, -k_y)$. Thus time reversal symmetry in this system is broken by the field $\mathbf{B}$.

These experiments are pertinent to quantum chaos because the electromagnetic wave equation in a thin microwave cavity with magnetized ferrite is in the same universality class (GUE) as the Schrödinger equation without time reversal symmetry. To be specific, in the presence of a time-independent applied magnetic field $\mathbf{B}_f = \nabla \times \mathbf{A}$ with the Coulomb gauge ($\nabla \cdot \mathbf{A} = 0$), the Schrödinger equation for a particle of mass m and charge q with wave function ψ in two dimensions (x, y) is

$$\nabla^2\psi - 2(iq/\hbar)\mathbf{A} \cdot \nabla\psi + (2m/\hbar^2) \times [E - (q^2/2m)\mathbf{A}^2]\psi = 0, \quad (2)$$

with $\psi = 0$ on the boundary. (This $\mathbf{B}_f$ should not be confused with the static magnetic field $\mathbf{B}$ used to magnetize the ferrite strip.) Equation (2) should be compared with the following electromagnetic wave equation in a microwave cavity with magnetized ferrite:

$$\nabla \cdot [(1 + \mu_{\|})\nabla E_z] - i(\hat{\mathbf{z}} \times \nabla\kappa) \cdot \nabla E_z + k^2 E_z = 0, \quad (3)$$

with $E_z = 0$ on the boundary. In the case, with $\nabla\mu_{\|} = 0$, Eq. (2) and Eq. (3) give the same eigenvalue equation

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[6] if one identifies E_z with ψ and, up to constant factors, $\mathbf{B}_f$ with $\hat{\mathbf{z}}\nabla^2\kappa$. In our experiment, κ changes discontinuously from zero in the empty region of the cavity to its value inside the ferrite strip. Thus, $\mathbf{B}_f$ in the analogous Schrödinger problem is a "double layer" (i.e., the derivative of a delta function on the surface of the ferrite). Even with this rather singular magnetic field, the analogy to the Schrödinger equation is still not perfect in the experiment because $\mu_\parallel$ also changes discontinuously crossing the ferrite boundary. Nevertheless, the relevant point is that the magnetized ferrite problem and the magnetized Schrödinger problem are in the same (GUE) universality class.

The geometry of our microwave cavity is shown in the inset of Fig. 1 where the curved boundaries are circular arcs. In this geometry, all typical ray-trajectory orbits are chaotic and all periodic orbits are isolated. Although the ferrite is a lossy material compared to a good conductor like copper, its degradation of the Q factor of the cavity near the gyromagnetic resonance remains relatively small because of the small volume of ferrite employed.

The microwave signal is coupled to the cavity electrically through four very small holes drilled in the top plate of the cavity. The coupling is chosen to be as weak as possible so that shifting and broadening of the cavity frequency resonances due to the coupling is minimized. The eigenmodes of the cavity are measured using an HP 8510C vector network analyzer by locating resonance peaks in the transmission spectra between pairs chosen from the four holes. Since the thickness of our cavity is $d = 0.3125$ in., we could, in principle, perform our frequency sweep up to $f_{\max} = c/(2d) \sim 18.9$ GHz while ensuring that the eigenmodes obtained correspond only to the 2D TM modes of the cavity. However, in practice, due to the finite Q factor of the cavity (on the order of several thousand), we found that we could reliably identify only resonance peaks up to approximately 16 GHz. Furthermore, since we are interested in the semiclassical behavior of the system, our experiment will examine the statistics of eigenmodes in the regime where the wavelength is small compared to cavity size. Correspondingly, in our experiment, we consider only frequencies above 7 GHz, which corresponds to mode numbers above $\sim$200. Within this frequency range, we could identify up to 800 eigenmodes. In studying the GUE statistics, however, it is necessary to consider smaller frequency ranges since the ferrite properties are strongly frequency dependent, and κ in Eq. (1) may not be sufficiently large to achieve full GUE statistics far from the gyromagnetic resonant frequency of the ferrite. In general, we expect that the phase difference, $\Delta\phi(B) = |\phi(B,k_y) - \phi(B,-k_y)|$, is large enough to yield GUE statistics when it is at least two orders of magnitude greater than $\Delta k\lambda$, where Δk is the average spacing between modes in k space and λ is the wavelength of a given eigenmode. (Note that $\Delta k\lambda \to 0$ as the mode number goes to infinity. Thus, in this limit, the transition from GOE to GUE occurs abruptly for any $|B| > 0$. On the other hand, the transition is continuous when the mode number is finite [7].) Using the values of the ferrite parameters supplied by the manufacturer [8], we found that data only from the upper range (13.5–16 GHz) of our operating frequency span provide sufficiently large phase difference $\Delta\phi$ for GUE statistics. This poses a limit on the number of energy levels ($\sim$260) we used in calculating the GUE statistics. Despite this limitation, we will show below that the quantitative difference between GOE and GUE statistics for the cases with and without magnetic field can still be unambiguously observed.

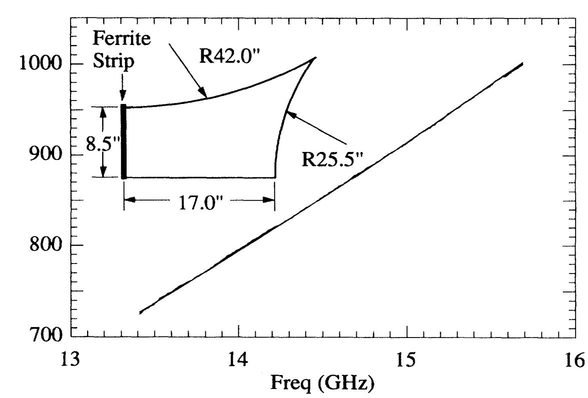

FIG. 1. Experimental cumulative level density $N(E)$ vs $f[E = (2\pi f/c)^2]$ in the case when the ferrite is magnetized. The theoretical curve for the smooth monotonic part $N_0(E)$ is superimposed on top. The inset shows the geometry of our microwave cavity.

In our experiment, the magnetic field is provided by a series of Nd-Fe-B magnets placed on both the top and the bottom plates of the cavity in an attracting position. These magnets are able to produce a field of approximately 2500 G in a 1-in. air gap. In general, the time irreversibility increases with the saturation magnetization of the ferrite. In our experiment, we have chosen a ferrite with a relatively high saturation magnetization ($4\pi M_s = 1850$ G) and a comparatively small resonance absorption, which is characterized by its resonance linewidth ($\Delta H = 14$ Oe). We note that the degree of time irreversibility can be adjusted by controlling the amount of the ferrite in the cavity, by changing the magnitude of the applied magnetic field, or by analyzing the data along different frequency windows of a fixed span. In this Letter, we will report experimental results with pure GUE statistics and on the GOE-GUE transition using different frequency windows of a fixed span. Results on the transition from GOE to GUE using the magnetic field as a varying parameter will be reported in a future paper. For theoretical accounts of this transition see Ref. [7].

As the first step in examining the spectral statistics of the experimental data, we construct the cumulative level density of states $N(E)$, which gives the number of "energy levels" with energy less than E [we identify E with $k^2 = (2\pi f)^2/c^2$ where f is the cavity resonant frequency]. From a semiclassical calculation, this "staircase" function consists of a smooth monotonic part and a fluctuating part, $N(E) = N_0(E) + N_{\rm fluc}(E)$. The smooth monotonic part, $N_0(E)$ to $O(E^{1/2})$ is given by [9] $N_0(E) = C_1 E + C_2 E^{1/2}$. In the case of the empty cavity without ferrite, $C_1 = A/4\pi$ where A is the cross-sectional area of the cavity; C_2 depends on the cavity boundary conditions: $C_2 = L/4\pi$ for Neumann boundary conditions and $C_2 = -L/4\pi$ for Dirichlet boundary conditions, where L is the perimeter of the cavity. In the semiclassical regime $[N_0(E) \gg 1]$, the first term, $C_1 E$, is large compared to the second term. Figure 1 is a graph of $N(E)$ vs f in the frequency range 13–16 GHz, for the case when the ferrite is magnetized by the applied magnetic field. We have also plotted $N_0(E)$ as a solid curve in which we use $C_2 = -L/4\pi$ and an area A that is 5% larger than the physical area of the cavity. This increase in area is meant to roughly account for the increased wave number in the ferrite: in the relevant frequency range, $k_f \delta / kb \approx 0.05$. Here, k_f is the wave number in the ferrite derived from the manufacturer's specifications; k is the vacuum wave number; δ is the ferrite thickness (0.09 in.); and b is the horizontal length of the cavity (17 in.). The fluctuating part, $N_{\rm fluc}(E)$, contains universal behavior which depends only on the symmetry class of the system. To examine these universal features, one makes a change of variables using $e = N_0(E)$, and defines the "unfolded" cumulative energy density as $\hat{N}(e) \equiv N(E)$.

Random matrix theory provides statistical predictions for the fluctuations of the unfolded energy spectrum. In particular, we concentrate on the *spectral rigidity* $\Delta(L)$ [10] and the *nearest neighbor spacing distribution* $P(s)$. To avoid using a histogram (which, because of our small number of levels, has large statistical fluctuations), we consider the integral $I(s) = \int_0^s P(s)\,ds$ rather than $P(s)$ itself. This allows us to estimate $I(s)$ from our data by simply counting the number of level spacings less than s, and dividing by the total number of spacings. The most interesting range is for small s where the level repulsion phenomena are distinctly different for GOE and GUE. In particular, the small s behavior is either quadratic or cubic: $I(s) \cong (\pi/4)s^2$ for GOE and $I(s) \cong [32/(3\pi^2)]s^3$ for GUE.

Our experimental results for $\Delta(L)$ and $I(s)$ are shown in Figs. 2, 3, and 4. As stated earlier, the effect of time reversibility is strongly frequency dependent. Experimentally, we have found that the effects of the ferrite are strongest in the range between 13.43 and 15.69 GHz, as expected from our numerical calculations of $\Delta\phi(B)$.

Figure 2 shows experimental plots of $\ln[I(s)]$ vs $\ln(s)$ for the case with no magnetic field (triangles) and with magnetic field (circles) in the GUE regime (13.43–15.69 GHz). The exact theoretical predictions are superimposed on top. We note that the agreement between these experimental plots and the theoretical curves is quite good. Moreover, for small s values, the best-fitted straight lines to $\ln[I(s)]$ give a slope of 2.02 for the time reversible case (GOE), as compared to the theoretical value of 2, and a slope of 2.88 for the time irreversible case (GUE), as compared to the theoretical value of 3.

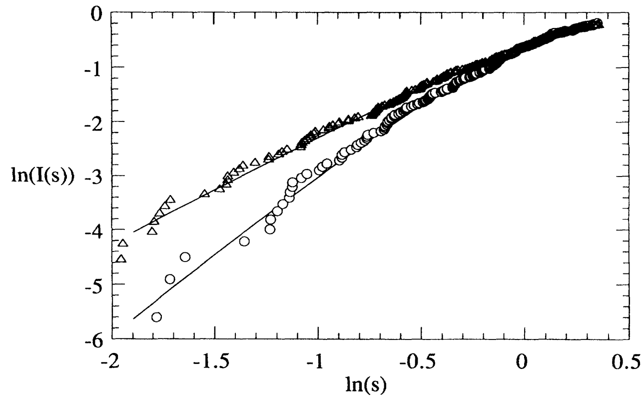

FIG. 2. Circles and triangles correspond to data with and without magnetic field, respectively. Data are from the range 13.43–15.69 GHz. Theoretical curves for the GOE and the GUE are superimposed.

Better statistical evidence for the GOE-GUE transition is provided by the $\Delta(L)$ plots in Fig. 3. Note that, since $\Delta(L)$ involves averaging over all the levels for each L, fluctuations are reduced, and a clear distinction between the two cases is evident. The lower solid curve of Fig. 3 is the GUE prediction for $\Delta(L)$ and the upper solid curve is the GOE prediction. Respectively, the diamonds and the crosses show data for the frequency range 13.43–15.69 GHz with and without the magnets in place. The

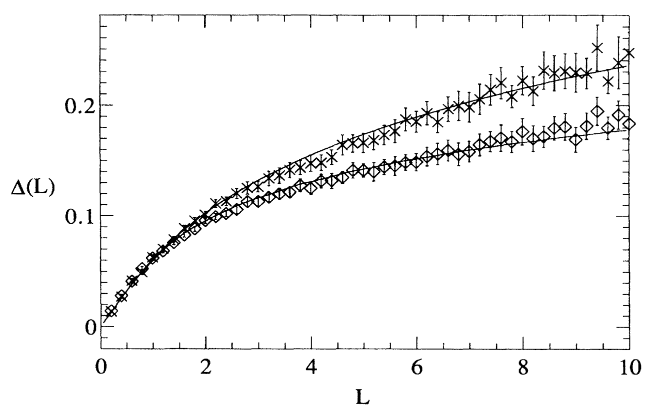

FIG. 3. Diamonds and crosses correspond to data with and without magnetic field, respectively. Data are from the range 13.43–15.69 GHz. Theoretical curves for the GOE (top solid curve) and the GUE (bottom solid curve) are superimposed. (These curves are from the exact integral expressions for $\Delta(L)$ (Ref. [10]).

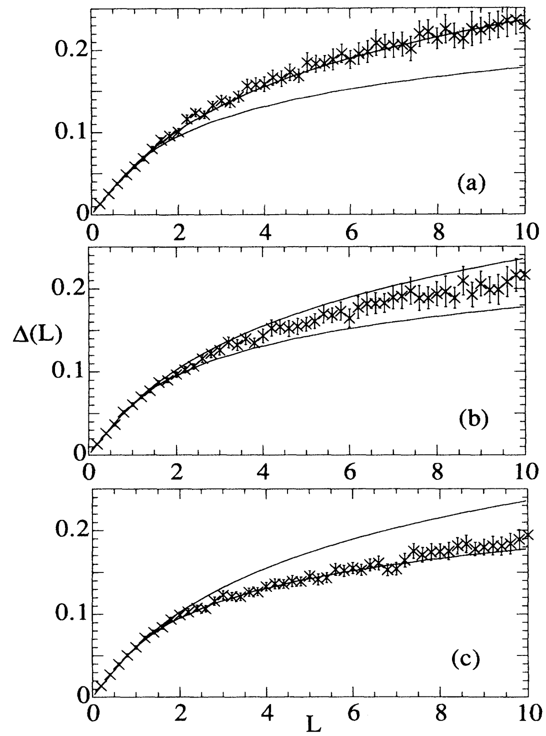

FIG. 4. $\Delta(L)$ vs L for three different frequency ranges with the static magnetic field held fixed. (a) 7–10.71 GHz, (b) 9.04–12.15 GHz, (c) 13.43–15.69 GHz. Theoretical curves for the GOE (top solid curve) and the GUE (bottom solid curve) are superimposed.

main qualitative effect of the time reversal symmetry breaking [viz. a decrease in $\Delta(L)$] is clearly evident in the data; and the agreement between the data and the solid curves is good.

Figure 4 gives a sense of the transition from GOE to GUE. The three $\Delta(L)$ plots in Fig. 4 are calculated with the static magnetic field held fixed and with approximately the same number of energy levels (~260). The frequency range used for Fig. 4(a) is 7–10.71 GHz. In this frequency range, we estimate from our calculations that $\Delta\phi(B,f)$ is not sufficiently large to alter the GOE statistics. In the frequency range, 13.43–15.69 GHz [Fig. 4(c)], we estimate that the difference $\Delta\phi(B,f)$ is sufficiently large to yield GUE statistics. The frequency range used in Fig. 4(b) (9.04–12.15 GHz) represents an intermediate case between GOE and GUE.

In conclusion, by placing a piece of magnetized ferrite inside a two-dimensional microwave cavity, we have successfully broken the time reversal symmetry of the system and have shown that the resultant energy spectrum agrees with the one predicted by the Gaussian unitary ensemble of random matrices [11]. Furthermore, by analyzing the data from different frequency ranges, we have experimentally observed the transition from GOE to GUE statistics.

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