Probability Amplitude Fluctuations in Experimental Wave Chaotic Eigenmodes with and Without Time-Reversal Symmetry

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We have measured the probability density $|\psi(r)|^2$ in the semiclassical limit of a classically chaotic square well potential with and without time reversal symmetry, and compared our findings with theoretical predictions. We find that wave functions with time-reversal symmetry have larger fluctuations than those without time-reversal symmetry. To quantify the degree of these fluctuations, eigenmodes both with and without time-reversal symmetry are statistically analyzed and the two-point spatial correlation function and the probability density distribution function of the eigenmodes are found to agree with theoretical predictions. [S0031-9007(98)07259-7]

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The quantum mechanical behavior of nonintegrable systems has been a very intriguing subject for many decades and is still one of active research [1]. This is not only because of the interesting fundamental physics of "quantum chaos" but also because of the strong analogy between the quantum mechanical behavior of mesoscopic systems and the wave chaotic behavior of nonintegrable systems [2]. Hence the subject can pave the way to understanding the statistical properties of electronic eigenstates and eigenfunctions of mesoscopic systems, such as quantum dots and quantum wires, which will be increasingly important for future technological applications.

Theories and numerical simulations suggest that for integrable (nonchaotic) systems a large degree of degeneracy is allowed in the eigenvalues of the system; therefore the spacing between neighboring eigenvalues obeys Poisson statistics. For nonintegrable systems the existence of classical chaos breaks the degeneracies, and therefore the eigenvalue spacing statistics are clearly no longer Poisson. Theories propose that the statistics of the eigenvalue spacing are governed not only by the integrability but also by the time-reversal symmetry of the system. One expects the spectral properties of a chaotic system with time-reversal symmetry to follow the statistics of a Gaussian orthogonal ensemble (GOE) of random matrices, while a chaotic system with broken time-reversal symmetry is expected to follow the statistics of a Gaussian unitary ensemble (GUE) of random matrices. In recent years several experiments similar to those described here have demonstrated that these theoretical predictions are indeed correct [3–5].

An even more intriguing subject is the investigation of the eigenfunctions of wave chaotic systems. Although some of the interesting behavior of eigenfunctions in systems with GOE symmetry has been explored by several groups [6–8], there has been no experimental examination of the time-reversal symmetry dependence of the eigenfunction behavior [9]. Investigating the detailed behavior of both time-reversal symmetric (TRS) and time-reversal symmetry broken (TRSB) wave functions is also imperative as it will give insights into the behavior of electronic wave functions of nanoscale structures that have these symmetries.

In this paper we report our experimental results on the behavior and statistics of eigenfunctions obtained from a quasi-two-dimensional (quasi-2D) microwave cavity analog of a quantum system with and without time-reversal symmetry. Our experiments indicate that the eigenmodes experience strong amplitude fluctuations in a classically chaotic potential well and the degree of fluctuations depends on the time-reversal symmetry. In our experiment the distribution of probability amplitude has an exponential dependence on $|\psi_n|^2$ for TRSB systems and the Porter-Thomas distribution for TRS systems (ψ_n is the eigenfunction with energy ϵ_n). The probability density correlation function obeys $\langle|\psi_n(\vec{r}_1)|^2|\psi_n(\vec{r}_2)|^2\rangle \propto 1 + c|f(\Delta r)|^2$, where $f(\Delta r)$ is the Friedel function, and $\Delta r = |\vec{r}_1 - \vec{r}_2|$. The factor c is predicted to be 1 for the TRSB case and 2 for the TRS case [9], in agreement with our experimental results.

To study the behavior of wave functions in a classically chaotic system with and without time-reversal symmetry, we employed quasi-2D microwave cavities and measured the microwave electric field strength inside the cavity [10]. For a quasi-2D system with TRS, the Helmholtz equation for the electric field, $\nabla^2 E_{z,n} + k_n^2 E_{z,n} = 0$, is mathematically identical to the Schrödinger equation for a particle of mass m in a potential V, $\nabla^2\psi_n + \frac{2m}{\hbar^2}(\epsilon_n - V)\psi_n = 0$, where $E_{z,n}$ is the electric field component perpendicular to the surface of the quasi-2D cavity and is analogous to the wave function ψ_n, $k_n^2 = \omega_n^2\mu\epsilon$, and ω_n is the nth resonant frequency of the cavity with uniform permittivity ϵ and permeability μ. To represent a quasi-2D system with TRSB, consider a homogeneously magnetized ferrite with an associated electromagnetic wave equation: $\nabla^2 E_{z,n} - i(\hat{z} \times \nabla\kappa) \cdot \nabla E_{z,n} + k^2 E_{z,n} = 0$, where κ is the off-diagonal component of the ferrite permeability tensor. This system is in the same universality class as the Schrödinger equation for a charged particle in a magnetic field [11]: $\nabla^2\psi_n - 2\frac{iq}{\hbar}\vec{A}\cdot\nabla\psi_n + \frac{2m}{\hbar^2}(\epsilon_n - V - \frac{q^2}{2m}\vec{A}^2)\psi_n = 0$, where $\vec{A}$ is the vector potential, and q is the charge of the particle [5,12]. In all cases $E_z = 0$ and $\psi_n = 0$ at the boundary of the potential well. The cavity supports 2D transverse magnetic modes (only E_z, B_x, and B_y are nonzero) below the 3D cutoff frequency, $f_{\max} = c/2d \simeq 18.9$ GHz, where $d = 0.310$ in. is the cavity thickness in the z direction.

Using these mathematical analogies, we examine the behavior of eigenmodes of a 2D cavity exhibiting classically chaotic motion for a billiard of the same geometry [10]. The geometry used for our experiment is a quadrant of a bow-tie-shaped region bounded by four circular arcs, two with a 42 in. radius and two with a 25.5 in. radius as shown in Fig. 1(a). This geometry was selected because all typical ray-trajectory orbits are chaotic and all periodic orbits are isolated.

For the investigation of time-reversal symmetry dependence, we place a thin ferrite strip with dimensions 8.4 in. $\times$ 0.2 in. $\times$ 0.310 in. adjacent to one wall of the cavity as shown in Fig. 1. For the TRSB case we apply a dc magnetic field to the ferrite and measure $|E_{z,n}|^2$ within certain frequency ranges where the ferrite gives a large nonreciprocal phase shift that breaks time-reversal symmetry [12]. For TRS behavior we also use a magnetized ferrite but only consider frequency ranges where the nonreciprocal phase shift due to the ferrite is very small.

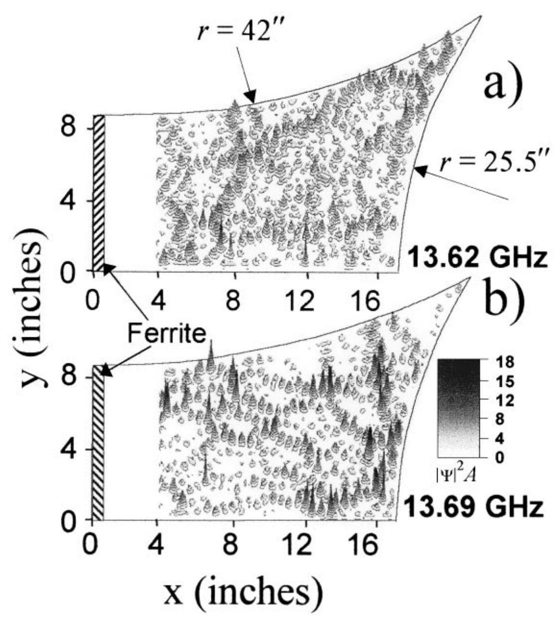

FIG. 1. Probability amplitude $|\psi_n|^2A$ as a function of position for eigenmodes of the "bow-tie" quadrant potential well. The eigenmodes have approximately the same frequency, 13.62 GHz (a) and 13.69 GHz (b), and are measured without (a) and with (b) time-reversal symmetry. The first 4 in. of each eigenmode are not imaged to prevent the perturbation scanning magnet from influencing the ferrite.

Microwaves are fed into the cavity through an optimally coupled dipole antenna in the top plate of the cavity, and we measure transmitted microwave signals through a similar antenna at another location on the top plate.

Wave function mapping is performed using a scanned perturbation method [10,13,14]. When a perturbation is introduced inside a cavity, the resonance frequency of the cavity changes as $\omega^2 = \omega_0^2[1 + \int(B^2 - E^2)dv_p]$, where ω_0 is an unperturbed resonance frequency of cavity. In this expression the microwave magnetic field B and the electric field E (which are normalized over the cavity volume) are integrated over the volume of the perturbation v_p. A small cylindrical metal pin (0.02 in. diameter, 0.16 in. long) is used as a perturbation and is kept aligned with the z axis while it is scanned by means of a carefully tailored magnet located outside the cavity [10]. Since this pin has a very small cross-sectional area in the horizontal plane, the magnetic field term is much smaller than the electric field term, particularly in comparison with commonly used spherical perturbations [10,14]. This simplifies the perturbed resonance frequency to $\omega^2 \approx \omega_0^2[1 - \int E_{z,n}^2 dv_p]$. In other words, $E_{z,n}^2$ at an arbitrary point on the x-y plane can be deduced by measuring the perturbed microwave resonant frequency. The normalized $E_{z,n}^2$ is therefore identical to $|\psi_n|^2$ for the quantum potential well that is analogous to the microwave cavity being studied. Our technique enables us to measure a wide dynamic range of $|\psi_n|^2$ for states in the semiclassical regime.

We have mapped over 200 images of different eigenmodes at frequencies between 700 MHz and 16 GHz [10,15]. Here we examine the eigenmodes between 10 and 16 GHz since we are interested in the high-lying states in the semiclassical regime where the wavelength of the quantum mechanical particle is much less than the characteristic size of the potential well. Figure 1 contrasts TRS and TRSB wave functions of nearly identical energy, with $\omega_0/2\pi \simeq 13.69$ and 13.62 GHz, respectively [15]. Figure 1(a) shows the probability amplitude, $|\psi_n|^2A$, pattern with TRSB, where A is the area of the cavity. The wave function exhibits irregular patterns and has much larger probability amplitude fluctuations than the rectangular wave functions we imaged [10]. The probability amplitude patterns with TRS, as shown in Fig. 1(b), reveal similar probability amplitude fluctuations; however, the TRS wave functions have larger probability amplitude fluctuations than the TRSB wave functions. These fluctuations are present in all probability amplitude patterns obtained from the bow-tie cavity. In addition, we find the TRSB probability amplitude peaks appear to be more smeared out over the 2D cavity surface compared to the TRS patterns.

To make a quantitative analysis of the degree of these probability amplitude fluctuations, we extract the distribution of the probability amplitude $P_0(v)$ from a histogram of the measured probability amplitude within a given wave function, where $v = |\psi_n|^2 A$. Figure 2 shows two density population distributions $P_0(v)$ which we obtained by averaging the probability density distributions of 9 individual eigenmodes for both the TRS and TRSB cases. $P_0(v)$ with TRS is consistent with the Porter-Thomas distribution [16], $P_0(v) = (1/\sqrt{2\pi v})\exp(-v/2)$, for GOE random matrix systems, shown as a thick solid line in Fig. 2. In contrast, $P_0(v)$ with TRSB is consistent with an exponential, $P_0(v) = \exp(-v)$, shown as a thin solid line, as theory predicts for GUE random matrix systems [9,16]. Furthermore, in Fig. 2, the probability density distributions, both with and without time-reversal symmetry, exhibit long tails. In contrast, for an integrable system it is known that $P_0(v)$ quickly dies off at a certain value of probability density [for example, $P_0(v) \simeq 0$ when $v \geq 4$ for a rectangular cavity] [17]. This means that chaotic systems allow larger fluctuations of probability density than integrable ones. Hence the distribution function suggests that strong wave function fluctuations are a signature of quantum systems in classically chaotic potentials. Also note from the distribution functions that wave chaotic eigenmodes with TRS have more large-amplitude fluctuations than TRSB eigenmodes with the same energy, as can be seen from the longer tail in the TRS distribution.

To cross-examine the degree of fluctuations we extract the variance, $\sigma^2 = \langle (v - \langle v \rangle)^2 \rangle$, of the $P_0(v)$ distribution functions for GOE and GUE symmetries. The variance of the two probability density distributions for the two symmetries is predicted to be $\int_0^\infty (v - \langle v \rangle)^2 P_0(v)\,dv = 2/\beta$ where β is the number of independent degrees of freedom in the random matrix elements. Random matrix theory indicates that $\beta = 1$ (real matrix elements) for TRS and $\beta = 2$ (complex matrix elements) for TRSB wave functions. The analysis of our data indicates that $\sigma^2_{\mathrm{TRSB}} = 1.10 \pm 0.14$ while $\sigma^2_{\mathrm{TRS}} = 1.61 \pm 0.09$. The TRSB result is consistent with theoretical predictions while the TRS result appears to be inconsistent with theory. However, when we take into account the finite experimental resolution for v $(0.05 \leq v \leq 10)$ [18], one finds $\sigma^2_{\mathrm{TRS,expt}} = \int_{0.05}^{10} (v - \langle v \rangle)^2 P_0(v)\,dv = 1.65$, in agreement with our data. Note that in each case the variance of the distribution is of the same order as the average, another indication that there are very strong fluctuations in the probability amplitude. We should also point out that while the Gaussian ensembles are discrete symmetries, our data suggest that there is a smooth transition between TRS ($\beta = 1$) and TRSB ($\beta = 2$) eigenfunctions due to the frequency dependence of the nonreciprocal phase shift from the ferrite [5,12]. Our data suggest that a general distribution, such as $P(v;\beta) = (\beta v/2)^{\beta/2-1} e^{-\beta v/2} (\beta/2)/\Gamma(\beta/2)$ [19–21], interpolates smoothly between the Porter-Thomas distribution (TRS) and a pure exponential (TRSB) and the parameter β may provide a measure of the time-reversal symmetry breaking of the system [22]. However, other distribution functions may describe the transition as well [23].

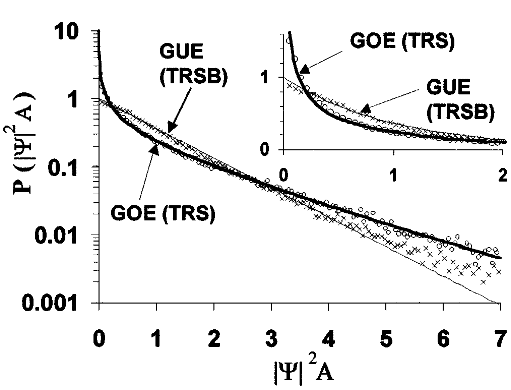

FIG. 2. Probability amplitude distribution of eigenmodes with (TRS) and without (TRSB) time-reversal symmetry. Also shown are theoretical predictions for systems with (GOE) and without (GUE) time-reversal symmetry, shown as thick and thin solid lines, respectively. The inset shows small probability amplitude behavior.

The two-point correlation of probability amplitudes in a given eigenfunction provides another test of the time-reversal symmetry of the system. Berry's intuitive derivation of the two-point correlation function for a wave chaotic eigenfunction is based on the conjecture that the wave function is an infinite superposition of plane waves having random amplitudes and directions but a fixed wave number. This assumption leads to the following expression for the angle-averaged two-point correlation [24]: $C(k\Delta r) = \langle |\psi_n(\vec{r}_1)|^2 |\psi_n(\vec{r}_2)|^2 \rangle \propto 1 + c|J_0(k\Delta r)|^2$, where the prefactor c depends on the time-reversal symmetry of the Hamiltonian, J_0 is Bessel's function of order zero, k is the wave number, and $\Delta r = |\vec{r}_1 - \vec{r}_2|$. The theoretical prediction is that $c = 2$ for GOE (TRS) and $c = 1$ for GUE (TRSB) [9]. The two-point correlation functions, which are calculated for each eigenfunction and then averaged to form the data shown in Fig. 3, are in good agreement with theory for the TRS (thick lines) and TRSB (thin lines) cases. If instead we average the best-fit c values for each eigenfunction our result for TRSB is $c = 1.11 \pm 0.12$ (averaged over 9 typical TRSB eigenfunctions) and for TRS $c = 2.06 \pm 0.11$ (averaged over 17 typical TRS eigenfunctions). As with the probability density distribution we find that between the TRS and TRSB regimes there exists a "crossover" region where the value of c goes smoothly from 2 to 1 [22].

In addition to good agreement with theory, we see qualitative features in the eigenmodes themselves that reinforce our findings. The TRS wave function depicted in Fig. 1(b) clearly has more large probability amplitude peaks than the TRSB wave function in Fig. 1(a). These peaks translate into a longer tail in the $P_0(v)$ distribution and a larger variance of $P_0(v)$ both theoretically predicted and confirmed in our experiment. Additionally TRS

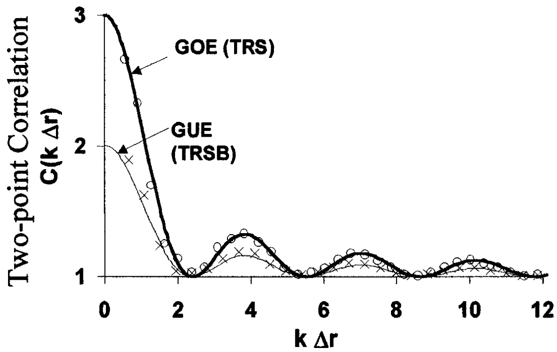

FIG. 3. Two-point correlation function of eigenmodes with (TRS) and without (TRSB) time-reversal symmetry, indicating the degree of spatial order in the wave functions. Theoretical predictions for systems with (GOE) and without (GUE) time-reversal symmetry are shown as thick and thin solid lines, respectively.

eigenfunctions have a large percentage of the probability amplitude concentrated into a small percentage of the total cavity area, while for TRSB eigenfunctions the probability amplitude is much more uniformly spread out. This can be seen from the smaller oscillations in the two-point correlation function for the TRSB eigenfunction shown in Fig. 3. The magnitude of these oscillations is inversely proportional to the spatial uniformity of the eigenmode, and in the limit of a completely homogeneous wave function, we would obtain a flat two-point correlation, $C(k\Delta r) = 1$.

In conclusion, our experimental results agree well with theoretical predictions for both the two-point correlation function and the probability density distribution for wave chaotic eigenfunctions both with time-reversal symmetry and time-reversal symmetry broken, and that between the two discrete symmetries is a continuous crossover region. Qualitatively, we find that eigenfunctions in wave chaotic systems which preserve time-reversal symmetry have stronger amplitude fluctuations and spatial correlations than their time-reversal symmetry broken counterparts in the semiclassical regime. These results on the statistics of eigenfunctions dovetail with our previous work on the eigenvalue spacing statistics of similar wave chaotic systems with and without time-reversal symmetry, and reinforce the conclusion that wave chaotic systems are very sensitive to time-reversal symmetry [5].

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