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Transmission Lines – Basic Principles

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TRANSMISSION LINES – BASIC PRINCIPLES

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1 Introduction

The aim of this lecture is to revise the basic principles of transmission lines in preparation for many of the lectures, which follow in this course. Obviously this lecture cannot cover such a wide topic in any depth and at the end of these notes are listed some textbooks which may prove useful for those wishing to go further into the subject.

Microwave measurements involve transmission lines because many of the circuits used are larger than the wavelength of the signals being measured. In such circuits, the propagation time for the signals is not negligible as it is at lower frequencies. So some knowledge of transmission lines is essential before sensible measurements can be made at microwave frequencies.

For many of the transmission lines, like coaxial cable and twisted pair lines, there are two separate conductors separated by an insulating dielectric. These lines can be described using voltages and currents in an equivalent circuit. However, another group of transmission lines, often called waveguides, like metallic waveguide and optical fibre, have no equivalent circuit and these are described in terms of their electric and magnetic fields. These notes will describe the two conductor transmission lines first, followed by a description of waveguides. The notes will end with some general

comments about attenuation, dispersion and power. A subsequent lecture will describe the properties of some of the transmission lines in common use today.

2 Loss-less Two Conductor Transmission Lines – Equivalent Circuit and Velocity of Propagation

All two-conductor transmission lines can be described using a distributed equivalent circuit. In order to simplify the treatment, the lines with no losses will be considered first. The lines have an inductance per meter, L , because the current going along one conductor and returning along the other produces a magnetic flux between the wires. Normally, at high frequencies the skin effect reduces the self-inductance of the wires to zero so that only this 'loop' inductance is important. The wires will also have a capacitance per meter, C , because any charges on one conductor will induce equal and opposite charges on the other. This capacitance between the wires is the dominant term and is much larger than any self-capacitance. The equivalent circuit is shown in figure 1.

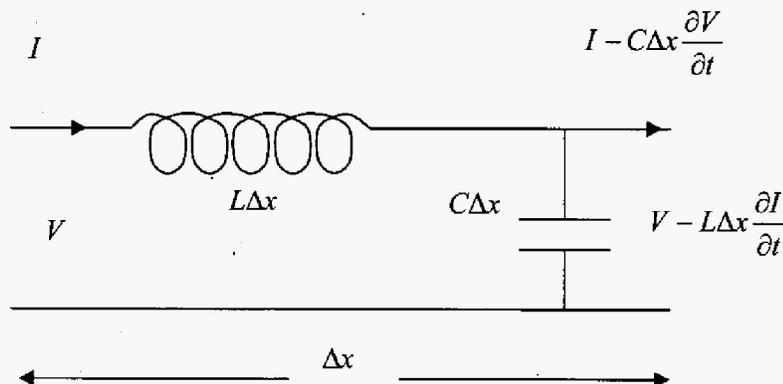


Figure 1: The equivalent circuit of a short length of transmission line with no losses.

If a voltage, V , is applied to the left hand side of the equivalent circuit, the voltage at the right hand side will be reduced by the voltage drop across the inductance. In mathematical terms:

V becomes $V - L\Delta x \frac{\partial I}{\partial t}$ in a distance Δx . So the change ΔV in that distance is given by:-

$$\Delta V = -L\Delta x \frac{\partial I}{\partial t}$$

Hence
$$\frac{\Delta V}{\Delta x} = -L \frac{\partial I}{\partial t}$$

and Lim
$$\frac{\Delta V}{\Delta x} = \frac{\partial V}{\partial x} = -L \frac{\partial I}{\partial t}$$

$$\Delta x \rightarrow 0$$

In a very similar fashion the current, I , entering the circuit on the left hand side is reduced by the small current going through the capacitor. Again, in mathematical terms:-

$$I \text{ becomes } I - C\Delta x \frac{\partial V}{\partial t} \text{ in a}$$

distance Δx .

So the change ΔI in that distance is given by:-

$$\Delta I = -C\Delta x \frac{\partial V}{\partial t}$$

Hence
$$\frac{\Delta I}{\Delta x} = -C \frac{\partial V}{\partial t}$$

and Lim
$$\frac{\Delta I}{\Delta x} = \frac{\partial I}{\partial x} = -C \frac{\partial V}{\partial t}$$

$$\Delta x \rightarrow 0$$

These equations are called the Telegraphists' equations:-

$$\frac{\partial V}{\partial x} = -L \frac{\partial I}{\partial t}$$

$$\frac{\partial I}{\partial x} = -C \frac{\partial V}{\partial t}$$

Differentiating these equations with respect to both x and t gives:-

$$\frac{\partial^2 V}{\partial x^2} = -L \frac{\partial^2 I}{\partial x \partial t}; \quad \frac{\partial^2 I}{\partial x^2} = -C \frac{\partial^2 V}{\partial x \partial t}$$

$$\frac{\partial^2 V}{\partial t \partial x} = -L \frac{\partial^2 I}{\partial t^2}; \quad \frac{\partial^2 I}{\partial t \partial x} = -C \frac{\partial^2 V}{\partial t^2}$$

Given that x and t are independent variables then the order of the differentiation is not important, the equations can be reformed into wave equations.

$$\frac{\partial^2 V}{\partial x^2} = LC \frac{\partial^2 I}{\partial t^2}; \quad \frac{\partial^2 I}{\partial x^2} = LC \frac{\partial^2 V}{\partial t^2}$$

The equations have general solutions of the form of any function of the variable $(t \mp \frac{x}{v})$.

So if any signal, which is a function of time, is introduced at one end of a loss-less transmission line then at a distance x down the line the function will be delayed by $\frac{x}{v}$. If the signal were travelling in the opposite direction the delay would be the same except x would be negative and the positive sign in the variable would be needed.

If the function is $f(t - \frac{x}{v})$ substituting in the wave equation gives:-

$$\frac{1}{v^2} f''(t - \frac{x}{v}) = LC f''(t - \frac{x}{v})$$

This shows that for all types of signal - pulse, triangular, sinusoidal - there is a unique velocity on loss-less lines, v , given by

$$v = \frac{1}{\sqrt{LC}}$$

This is the velocity of both the current and voltage waveforms since the same wave equation governs both parameters.

2.1 Characteristic Impedance

The relationship between the voltage waveform and the current waveform is derived from the Telegraphists' equations.

If the voltage waveform is:-

$$V = V_0 f\left(t - \frac{x}{v}\right)$$

then
$$\frac{\partial V}{\partial x} = -\frac{V_0}{v} f'\left(t - \frac{x}{v}\right)$$

Using the first Telegraphist equation:-

$$\frac{\partial I}{\partial t} = \frac{V_0}{Lv} f'\left(t - \frac{x}{v}\right)$$

Integrating with respect to time gives:-

$$I = \frac{V_0}{Lv} f\left(t - \frac{x}{v}\right) = \frac{V}{Lv}$$

$$\text{So } \frac{V}{I} = Lv = \sqrt{\frac{L}{C}}$$

This ratio is called the characteristic impedance Z_0 and for loss-less lines.

$$\frac{V}{I} = Z_0$$

This is for waves travelling in a positive x direction. If the wave was travelling in a negative x direction, i.e. a reverse or backward wave, then the ratio of V to I would be equal to $-Z_0$.

2.2 Reflection Coefficient

A transmission line may have at its end an impedance, Z_L , which is not equal to the characteristic impedance of the line Z_0 . Thus, a wave on the line faces the dilemma of obeying two different Ohm's laws. In order to achieve this a reflected wave is formed. Giving positive suffices to the incident waves and negative suffices to the reflected waves the Ohm's law relationships become:

$$\frac{V_+}{I_+} = Z_0$$

$$\frac{V_-}{I_-} = -Z_0$$

$$\frac{V_L}{I_L} = \frac{V_+ + V_-}{I_+ + I_-} = Z_L$$

where V_L and I_L are the voltage and current in the terminating impedance Z_L .

A reflection coefficient, ρ or Γ , is defined as the ratio of the reflected to the incident wave. Thus:

$$\Gamma = \frac{V_-}{V_+} = -\frac{I_-}{I_+}$$

$$\text{So } Z_L = \frac{V_+(1+\Gamma)}{I_+(1-\Gamma)} = \frac{Z_0(1+\Gamma)}{1-\Gamma}$$

$$\text{And } \Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Since Z_0 for loss-less lines is real and Z_L may be complex then Γ in general will be complex. One of the main parts of microwave impedance measurement is to measure the value of Γ and hence Z_L .

3.2 Phase Velocity and Phase Constant for Sinusoidal Waves

So far the treatment has been perfectly general for shapes of waves. In this section, just the sine waves will be considered. In figure 2, a sine wave is shown at one instant in time. Since the waves move down the line with a velocity of v , the phase of the waves further down the line will be delayed compared with the phase of the oscillator on the left hand side of figure 2.

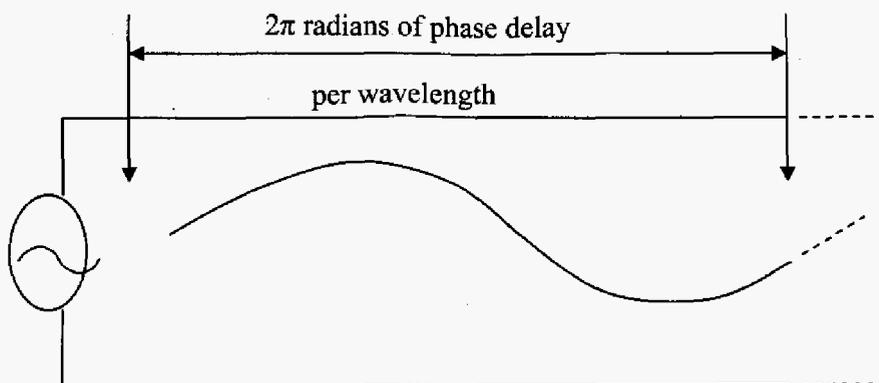


Figure 2: Sine waves on transmission lines

The phase delay for a whole wavelength is equal to 2π . The phase delay per metre is called β and is given by:-

$$\beta = \frac{2\pi}{\lambda}$$

Multiplying top and bottom of the right hand side by frequency gives:-

$$\beta = \frac{2\pi f}{\lambda f} = \frac{\omega}{v} = \omega\sqrt{LC}$$

Where v is now the phase velocity, ie the velocity of a point of constant phase and is the same velocity that is given in Section 2 if the lines are lossless.

2.4 Power Flow for Sinusoidal Waves

If a transmission line is terminated in an impedance equal to Z_0 then all the power in the wave will be dissipated in the matching terminating impedance. For lossless lines a sinusoidal wave with an amplitude V_1 the power in the termination would be:-

$$\frac{V_1^2}{2Z_0}$$

This is also the power in the wave arriving at the matched termination.

If a transmission line is not matched then part of the incident power is reflected (see Section 2.2) and if the amplitude of the reflected wave is V_2 then the reflected power is

$$\frac{V_2^2}{2Z_0}$$

$$\text{Since } |\Gamma| = \frac{V_1}{V_2}$$

$$|\Gamma|^2 = \frac{\text{Then Power reflected}}{\text{Incident Power}}$$

Clearly, for a good match the value of $|\Gamma|$ should be near to zero. The return loss is often used to express the match:-

$$\text{Return loss} = 10 \log_{10} \frac{1}{|\Gamma|^2}$$

In microwave circuits a return loss of greater than 20dB means that less than 1% of the incident power is reflected.

Finally, the power transmitted into the load is equal to the incident power minus the reflected power. A transmission coefficient, τ , is used as follows:-

$$\frac{\text{Transmitted Power}}{\text{Incident Power}} = |\tau|^2 = 1 - |\Gamma|^2$$

2.5 Standing Waves resulting from Sinusoidal Waves

When a sinusoidal wave is reflected by a terminating impedance which is not equal to Z_0 , the incident and reflected waves form together a standing wave.

If the incident wave is:-

$$V_+ = V_1 \sin(\omega t - \beta x)$$

and the reflected wave is:

$$V_- = V_2 \sin(\omega t + \beta x)$$

where $x=0$ at a distance D from the termination.

Then at some points on the line the two waves will be in phase and the voltage will be:-

$$V_{MAX} = (V_1 + V_2)$$

where V_{MAX} is the maximum of the standing wave pattern. At other points on the line the two waves will be out of phase and the voltage will be

$$V_{MIN} = (V_1 - V_2)$$

V_{MIN} is the minimum of the standing wave pattern. The Voltage Standing Wave Ratio or VSWR or S is defined as:-

$$S = \frac{V_{MAX}}{V_{MIN}}$$

Now

$$S = \frac{V_1 + V_2}{V_1 - V_2} = \frac{V_1(1 + |\Gamma|)}{V_1(1 - |\Gamma|)}$$

$$\text{So } S = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

$$\text{Or } |\Gamma| = \frac{S - 1}{S + 1}$$

Measuring S is relatively easy and so a value for $|\Gamma|$ can be obtained. From the position of the maxima and minima the argument or phase of Γ can be found. For instance, if a minimum of the standing wave pattern occurs a distance D from a termination then the phase difference between the incident and reflected waves at that point must be $n\pi$ ($n = 1, 3, 5 \dots$). Now the phase delay as the incident wave goes from that point to the termination is βD . The phase change on reflection is the argument of Γ . Finally, the further phase delay as the reflected wave travels back to D is also βD . So:-

$$n\pi = 2\beta D + \arg(\Gamma)$$

So, by a measurement of D and a knowledge of β the phase of Γ can also be measured.

3 Two Conductor Transmission Lines with Losses. Equivalent Circuit and Low-loss Approximation.

In many two-conductor transmission lines there are two sources of loss which cause the waves to be attenuated as they travel along the line. One source of loss is the ohmic resistance of the conductors. This can be added to the equivalent circuit by using a distributed resistance, R , whose units are ohms per metre. Another source of loss is the ohmic resistance of the dielectric between the lines. Since this is in parallel with the capacitance it is usually added to the equivalent circuit using a distributed conductance, G , whose units are Siemens per metre. The full equivalent circuit is shown in figure 3.

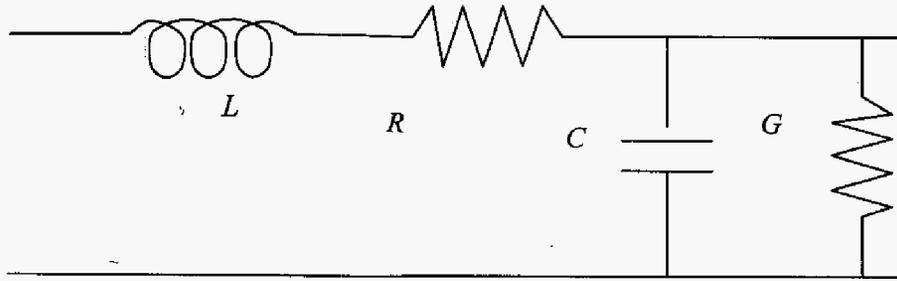


Figure 3: The equivalent circuit of a line with losses.

The Telegraphists' equations become:

$$\frac{\partial V}{\partial x} = -L \frac{\partial I}{\partial t} - RI$$

$$\frac{\partial I}{\partial x} = -C \frac{\partial V}{\partial t} - GV$$

Again, wave equations can be found by differentiating with respect to both x and t .

$$\frac{\partial^2 V}{\partial x^2} = LC \frac{\partial^2 V}{\partial t^2} + (LG + RC) \frac{\partial V}{\partial t} + RGV$$

$$\frac{\partial^2 I}{\partial x^2} = LC \frac{\partial^2 I}{\partial t^2} + (LG + RC) \frac{\partial I}{\partial t} + RGI$$

These equations are not easy to solve in the general case. However, for sinusoidal waves on lines with small losses, i.e. $\omega L \gg R$; $\omega C \gg G$ there is a solution of the form: -

$$V = V_0 \exp(-\alpha x) f\left(t - \frac{x}{v}\right)$$

where $v = \frac{1}{\sqrt{LC}}$ ms^{-1} the same as for loss-less lines: -

$$\alpha = \frac{R}{2Z_0} + \frac{GZ_0}{2} \text{ nepers m}^{-1}$$

$$\text{or } \alpha = 8.686 \left[\frac{R}{2Z_0} + \frac{GZ_0}{2} \right] \text{ dB m}^{-1}$$

$$\beta = \omega \sqrt{LC} \text{ radians m}^{-1} \text{ as for loss-less lines}$$

$$Z_0 = \sqrt{\frac{L}{C}} \text{ as for loss-less lines}$$

3.1 Pulses on transmission lines with losses

As well as attenuation a pulse on a transmission line with losses will also change its shape. This is caused by the fact that all the components of the transmission lines L , C , G and R are actually different functions of frequency. So, if the sinusoidal components of the pulse are considered separately they all travel at different velocities and with different attenuation. This frequency dependence is called dispersion. For a limited range of frequencies it is sometimes possible to describe a group velocity which is the velocity of the pulse rather than the velocity of the individual sine waves that make up the pulse. One effect of dispersion in pulses is the rise time is reduced and often the pulse width is increased. It is beyond the scope of these notes to include a more detailed treatment of this topic.

3.2 Sinusoidal Waves on Transmission Lines with Losses

For sinusoidal waves there is a solution of the wave equation and it is: -

$$\alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$$

and
$$Z_o = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

In general α , β and Z_o are all functions of frequency. In particular, Z_o at low frequencies can be complex and deviate considerably from its high frequency value. Since R , G , L and C also vary with frequency, a careful measurement of these properties at each frequency is required to characterise completely the frequency variation of Z_o .

4. Lossless Waveguides

These transmission lines cannot be easily described in terms of voltage and current as they sometimes only have one conductor, e.g. metallic waveguide or no conductor, e.g. optical fibre. The only way to describe their electrical properties is in terms of the electromagnetic fields that exist in, and in some cases, around their structure. This section of the notes will begin with a revision of the properties of a plane or transverse electromagnetic (T.E.M.) wave. The characteristics of metallic waveguides will then be described using these waves. The properties of other waveguiding structures will be given in a later lecture.

4.1 Plane (or Transverse) Electromagnetic Waves

A Plane (or Transverse) Electromagnetic Wave has two fields which are perpendicular or transverse to the direction of propagation. One of the fields is the electric field and the direction of this field is usually called the direction of polarisation (e.g. vertical, horizontal, etc.). The other field which is at right angles to both the electric field and the direction of propagation is the magnetic field. These two fields together form the electromagnetic wave. The electromagnetic wave equations for waves propagating in the z direction are: -

$$\frac{\partial^2 E}{\partial z^2} = \mu\epsilon \frac{\partial^2 E}{\partial t^2}$$

$$\frac{\partial^2 H}{\partial z^2} = \mu\epsilon \frac{\partial^2 E}{\partial t^2}$$

where μ is the permeability of the medium. If μ_R is the relative permeability then $\mu = \mu_R \mu_o$ and μ_o is the free space

permeability and has a value of $4\pi \cdot 10^{-7} \text{ Hm}^{-1}$. Similarly, ϵ is the permittivity of the medium and if ϵ_R is the relative permittivity then $\epsilon = \epsilon_R \epsilon_o$ and ϵ_o is the free space permittivity and has a value of $8.854 \cdot 10^{-12} \text{ Fm}^{-1}$. These wave equations are analogous to those in section 2 of these notes. The variables V , I , L and C are replaced with the new variables E , H , μ and ϵ and the same results follow. For a plane wave the velocity if the wave in the z direction is v given by: -

$$v = \frac{1}{\sqrt{\mu\epsilon}} \quad (\text{see Section 2 where } v = \frac{1}{\sqrt{LC}})$$

If $\mu = \mu_o$ and $\epsilon = \epsilon_o$, then

$$v_o = 2.99792458 \cdot 10^8 \text{ ms}^{-1}$$

The ratio of the amplitude of the electric field to the magnetic field is called the intrinsic impedance and has the symbol η .

$$\eta = \sqrt{\frac{\mu}{\epsilon}} = \frac{E}{H} \quad (\text{see Section 2 where } Z_o = \sqrt{\frac{L}{C}})$$

If $\mu = \mu_o$ and $\epsilon = \epsilon_o$, then $\eta_o = 376.61\Omega$ or $120\pi\Omega$

As in Section 2 fields propagating in the negative z direction are related by $-\eta$. The only difference is the orthogonality of the fields, which comes from Maxwell's Equations. For an electric field polarised in the x direction: -

$$\frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t}$$

If E_x is a function of $\left(t - \frac{z}{v}\right)$ as before

then

$$\frac{\partial E_x}{\partial z} = -\frac{1}{v} E_o f' \left(t - \frac{z}{v}\right) = -\mu \frac{\partial H_y}{\partial t}$$

$$H_y = \frac{1}{\mu v} E_o f \left(t - \frac{z}{v}\right) = \frac{1}{\mu v} E_x$$

$$\text{So } \frac{E_x}{H_y} = \mu v = \sqrt{\frac{\mu}{\epsilon}} = \eta$$

If an E_y field was chosen, the magnetic field would be in the negative x direction. For sinusoidal waves the phase constant is called the wave number and given symbol k .

$$k = \omega\sqrt{\mu\epsilon} \quad (\text{see Section 2 where } \beta = \omega\sqrt{LC})$$

Inside all two conductor transmission lines are various shapes of plane waves and it is possible

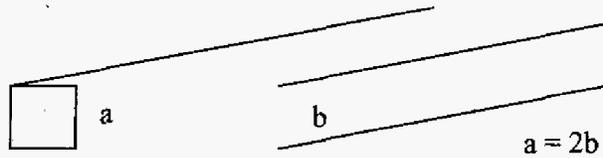


Figure 4: A rectangular metallic waveguide

If a plane wave enters the waveguide such that its electric field is in the y (or vertical) direction and its direction of propagation is not in the z direction it will be reflected back and forth by the metal walls in the y direction. Each time the wave is reflected it will have the phase reversed so that the sum of the electric fields on the surfaces of the two walls in the y direction is zero. This is consistent with the walls being metallic and therefore, good conductors and capable of short circuiting any electric fields.

to describe them completely in terms of fields rather than voltages and currents. The electromagnetic wave description is more fundamental but the equivalent circuit description is often easier to use. At high frequencies, two conductor transmission lines also have higher order modes and the equivalent circuit model for these becomes more awkward to use whereas the electromagnetic wave model is able to accommodate all such modes.

4.2 Rectangular Metallic Waveguides

Figure 4 shows a rectangular metallic waveguide.

The walls in the x direction are also good conductors but are able to sustain these electric fields perpendicular to their surfaces. Now, if the wave after two reflections has its peaks and troughs in the same positions as the original wave then the waves will add together and form a mode. If there is a slight difference in the phase the vector addition after many reflections will be zero and no mode is formed. The condition for forming a mode is thus a phase condition and it can be found as follows.

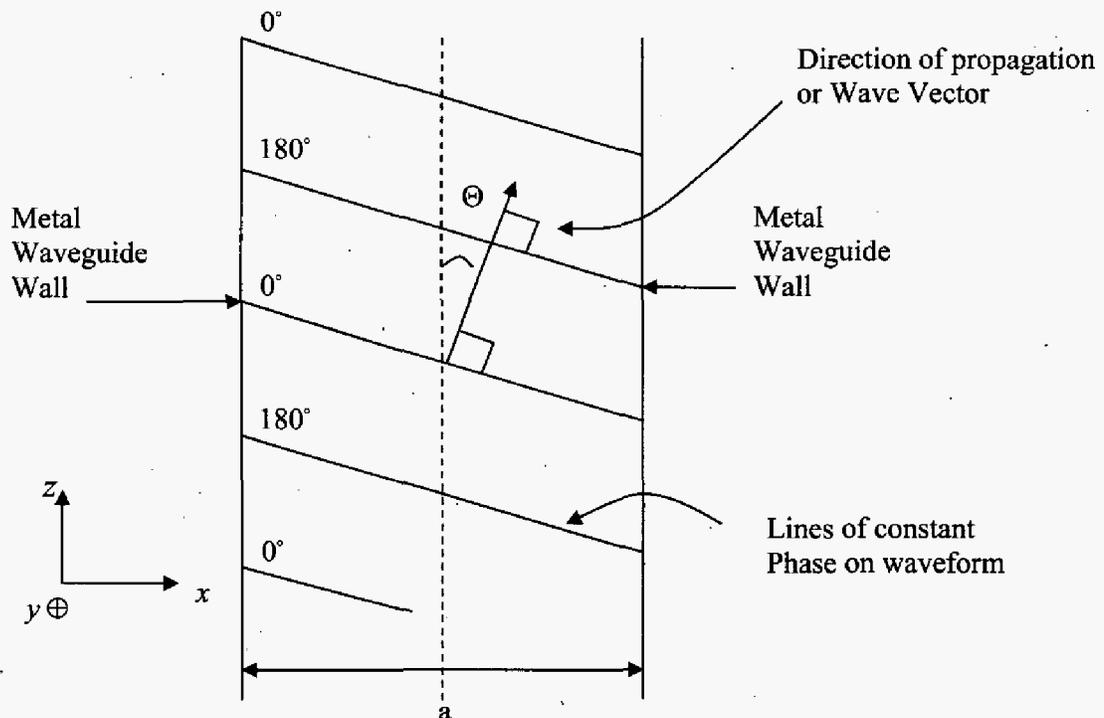


Figure 5: A plane wave in a rectangular metallic waveguide

Figure 5 shows a plane wave in a rectangular metallic waveguide with its electric field in the y direction and the direction of propagation at an angle Θ to the z direction.

As this wave is incident on the right wall of the waveguide in figure 5, it will give reflection according to the usual laws of reflection as shown in figure 6.

The rate of change of phase in the direction of propagation is k_0 , the free space wave number.

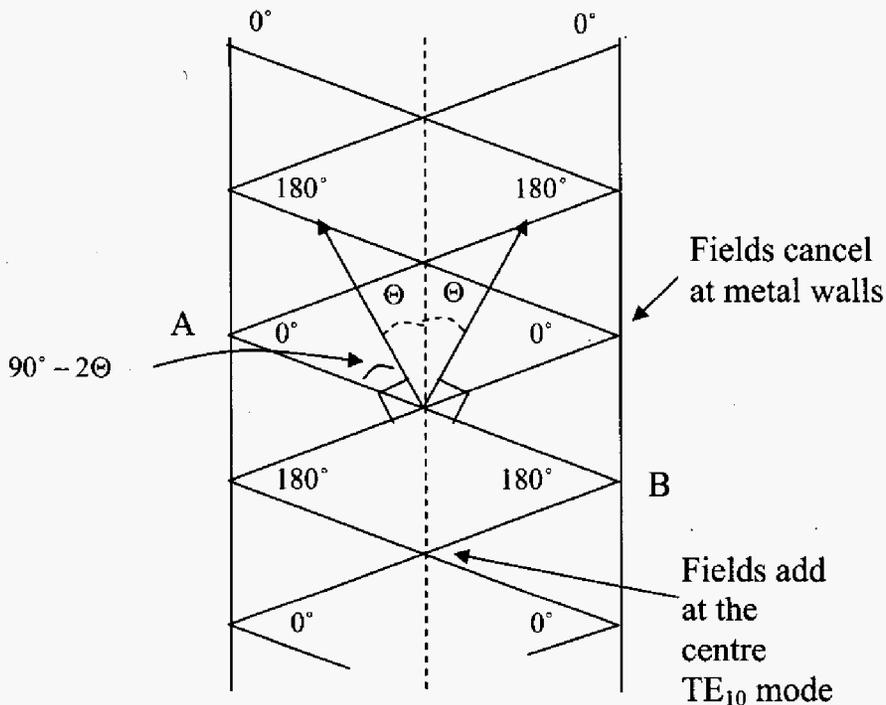


Figure 6: Two plane waves in a rectangular metallic waveguide. The phases 0° and 180° refer to the lines below each figure.

On further reflection, to form a mode this wave must 'rejoin' the original wave. So, figure 6 also shows the sum of all the reflections forming two waves one incident on the right wall one on the left. The waves form the mode if they are linked together in phase.

where $m = 0, 1, 2, 3$ etc.

The phase delay can be found from resolving the wave number along BA. This is :-

Consider the line AB. This is a line of constant phase for the wave moving to the right. Part of that wave at B reflects and moves along BA to A where it reflects again and rejoins the wave with the same phase. At the first reflection there is a phase shift of π . Then along BA there is a phase delay followed by another phase shift of π at the second reflection. The phase condition is :-

$$k_0 \sin 2\Theta \quad \text{radians m}^{-1}$$

If the walls in the y direction are separated by a distance a, then: -

$$AB = \frac{a}{\cos \Theta}$$

$2\pi +$ phase delay along BA for wave moving to the left = $2m\pi$

So the phase delay is $k_0 \sin 2\Theta \frac{a}{\cos \Theta}$ or $2k_0 \sin \Theta$

Hence the phase condition is:

$$k_o a \sin \Theta = m\pi \quad \text{where } m = 0, 1, 2, 3 \dots$$

The solutions to this phase condition give the various waveguide modes for waves with fields only in the y direction, i.e. the TE_{m0} modes. 'm' is the number of half sine variations in the x direction.

4.3 The Cut-off Condition

From the phase condition since $k_o = \frac{\omega}{v_o}$ then

$$\omega \sin \Theta = \frac{v_o m \pi}{a} \quad \text{is the phase condition.}$$

The terms on the right hand side are constant. For very high frequencies the value of Θ tends to be zero and the two waves almost propagate in the z direction. However, if ω reduces in value the largest value of $\sin \Theta$ is 1 and at this point the mode is cut off and can no longer propagate. The cut-off frequency is ω_c and is given by:-

$$\omega_c = \frac{v_o m \pi}{a}$$

or $f_c = \frac{v_o m}{2a}$ where f_c is the cut-off frequency.

If $v_o = \lambda_c f_c$ then

$$\lambda_c = \frac{2a}{m} \quad \text{where } \lambda_c \text{ is the cut-off wavelength.}$$

A simple rule for TE_{m0} modes is that at cut-off, the wave just fits in 'sideways'. Indeed, since $\Theta = 90^\circ$ at cut-off the two plane waves are propagating from side to side with a perfect standing wave between the walls.

4.4 The Phase Velocity

All waveguide modes can be considered in terms of plane waves. Since the simpler modes, just considered consist of just two plane waves they form a standing wave pattern in the x direction and yet form a travelling wave in the z direction. Since the phase velocity in the z direction is related to the rate of change of phase, i.e. the wave number then: -

$$\text{Velocity in the } z \text{ direction} = \frac{\omega}{\text{wave number in the } z \text{ direction}}$$

Using figure 5 or 6 the wave number in the z direction is

$$k_o \cos \Theta$$

$$\text{Now from the phase condition } \sin \Theta = \frac{m\pi}{k_o a}$$

$$\text{So } k_o \cos \Theta = k_o \left(1 - \left(\frac{m\pi}{k_o a} \right)^2 \right)^{\frac{1}{2}}$$

$$\text{Also } k_o = \frac{2\pi}{\lambda_o} = \frac{2\pi f}{\lambda_o} = \frac{\omega}{v_o}$$

Where v_o is the free space velocity

$$= \frac{1}{\sqrt{\mu_o \epsilon_o}}$$

Hence the velocity in the z direction

$$v_z = \frac{\omega}{k_o \cos \Theta}$$

$$v_z = \frac{v_o}{\left[1 - \left(\frac{m\lambda_o}{2a} \right)^2 \right]^{\frac{1}{2}}} = \frac{v_o}{\left[1 - \left(\frac{\lambda_o}{\lambda_c} \right)^2 \right]^{\frac{1}{2}}}$$

As can be seen from this condition when λ_o is equal to the cut-off wavelength (see Section 4.3) then v_z is infinite. As λ_o gets smaller than λ_c then the velocity approaches v_o . The phase velocity is thus always greater than v_o . Waveguides are not normally operated near cut-off as the high rate of change of velocity means impossible design criteria and high dispersion.

4.5 The Wave Impedance

The ratio of the electric to the magnetic field for a plane wave was discussed in Section 4.1 of these notes. Although the waveguide has two plane waves in it the wave impedance is defined

as the ratio of the transverse electric and magnetic fields. For TE modes this is given the symbol Z_{TE} .

$$Z_{TE_{mn}} = \frac{E_y}{H_x} = \frac{E_o}{H_o \cos \Theta} = \frac{\eta}{\cos \Theta}$$

where E_o and H_o refer to the plane waves. The electric fields of the plane waves are in the y direction but the magnetic fields are at an angle Θ to the x direction.

So

$$Z_{TE_{mn}} = \frac{\eta}{\left[1 - \left(\frac{m\lambda_o}{2a}\right)^2\right]^{\frac{1}{2}}} = \frac{\eta}{\left[1 - \left(\frac{\lambda_o}{\lambda_c}\right)^2\right]^{\frac{1}{2}}}$$

Thus, for the $\lambda_o \leq \lambda_c$ the value of $Z_{TE_{mn}}$ is always greater than η_o . A typical value might be 500Ω .

4.6 The Group Velocity

Since a plane wave in air has no frequency dependant parameters like those of the two conductor transmission lines, i.e. μ_o and ϵ_o are constant, then there is no dispersion and so the phase velocity is equal to the group velocity. A pulse in a waveguide therefore, would travel at v_o at an angle of Θ to the z axis. The group velocity along the z axis is given by

$$\text{Group velocity, } v_g = v_o \cos \Theta$$

$$= v_o \left(1 - \left(\frac{m\lambda_o}{2a}\right)^2\right)^{\frac{1}{2}}$$

The group velocity is always less than v_o and is a function of frequency.

For rectangular metallic waveguides: -

$$\text{Phase Velocity} \times \text{Group Velocity} = v_o^2$$

4.7 General Solution

In order to obtain all the possible modes in a rectangular metallic waveguide the plane wave must also have an angle ψ to the z axis in the y

z plane. This will involve the wave reflecting from all four walls. If the two walls in the x direction are separated by a distance b then the following are valid for all modes: -

$$\text{Let } A = \left[1 - \left(\frac{m\lambda_o}{2a}\right)^2 - \left(\frac{n\lambda_o}{2b}\right)^2\right]^{\frac{1}{2}}$$

$$m = 0, 1, 2,$$

$$n = 0, 1, 2,$$

$$\text{Then the velocity } v = \frac{v_o}{A}$$

$$Z_{TE} = \frac{\eta}{A}$$

$$v_g = Av_o$$

The modes with the magnetic field in the y direction - the dual of TE - are called Transverse Magnetic Modes, or TM. They have a constraint that neither m or n can be 0 as the electric field for these modes has to be zero at all four walls.

$$Z_{TM} = \eta A$$

The relative cut-off frequencies are shown in figure 7. As can be seen in that diagram mono-mode propagation using the TE_{10} mode is possible up to twice the cut-off frequency. However, the full octave bandwidth is not used as propagation near cut-off is difficult and just below the next mode can be hampered by energy coupling into that mode as well.

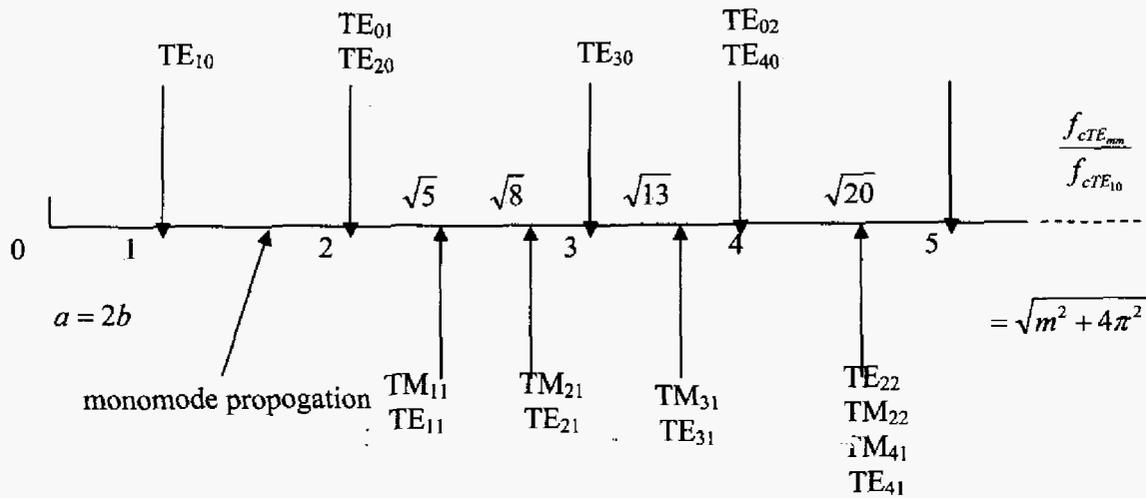


Figure 7: Relative cut-off frequencies for rectangular metallic waveguides

5. General References

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