A Hybrid Method for Quantitative Statistical Analysis of In-situ IC and Electronics in Complex and Wave-chaotic Enclosures

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Abstract— This research concerns with a quantitative statistical characterization of IC and electronic systems in large complicated enclosures. The objective is to identify, quantify, predict and characterize the in-situ performance of IC electronics housed inside the enclosure of interest, such as a room, aircraft fuselage, or computer box. A novel hybrid deterministic and stochastic formulation is proposed, in which small electronic components (circuits, packages, PCBs, etc.) in the computational domain are modeled using first-principles and large portions (cavity enclosures) are modeled statistically. The capability and benefits of the computation algorithms are exploited, illustrated and validated through representative product-level IC and electronic systems.

1. INTRODUCTION

Characterizing the integrated circuit (IC) and electronics within large complicated enclosures is an important problem with various applications [1–4]. A representative computer electronic system is shown in Fig. 1, in which a product-level IBM package [5] is integrated on a generic printed circuit board (PCB). The PCB, along with a monopole antenna and a mode stir, is inside a complicated cavity. As illustrated in Fig. 1, individual sub-systems exhibit vast differences in the aspect ratios (ratio of wavelength to feature size). Even with state-of-the-art full wave approaches, the computational resources required for such large multi-scale problems are prohibitively expensive. Furthermore, in high-frequency regime, electromagnetic (EM) wave solutions inside these enclosures show strong fluctuations that are extremely sensitive to the exact geometry of the enclosure, the location of internal electronics and the operating frequency. This phenomenon, known as wave or quantum chaos [6], has been discussed in the context of acoustics [7], electromagnetics [8–10], and quantum mechanics [11, 12]. In wave-chaotic systems, minor changes in the shape of the enclosure, or the reorientation of internal IC or electronics, can result in significantly different EM environments within the enclosure. Further, imprecise knowledge of these parameters is another obstacle to predictability. Therefore, "numerically exact" solutions obtained by a deterministic approach for a specific structure may be of limited practical value. It necessitates a quantitative statistical analysis of the in-situ IC performance and system behavior.

The research objective is to identify, quantify, predict and characterize the in-situ performance of IC electronics housed inside the enclosure of interest, such as a room, aircraft fuselage, or computer box. The main challenges are due to the computational complexity for extreme multi-scale computations accounting for mutual interactions of interconnects, packages, boards and systems,

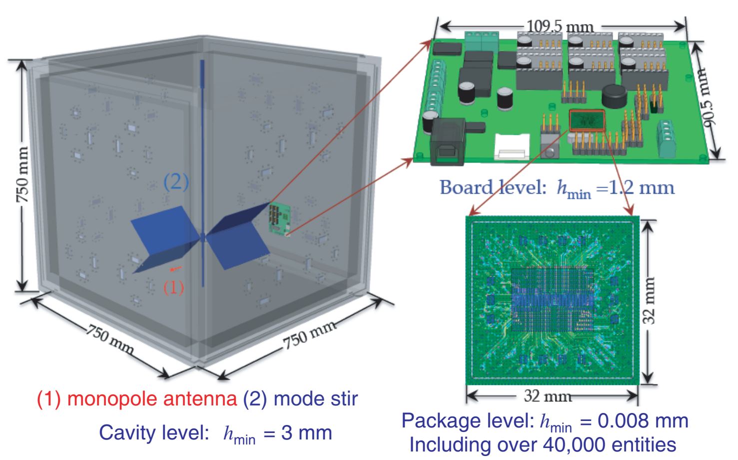


Figure 1: A complex electronic system from case, board to package level.

and the engineering need for a quantitative statistical characterization of IC and electronics in complex environments exhibiting wave chaos. In recent literature, non-overlapping and non-conformal domain decomposition (DD) methods [13, 14] have been proposed to address the geometrical complexity of ICs and packages. The electric and magnetic fields continuities across domain interfaces are enforced by the second order transmission condition [14], which leads to the scalable convergence of DD iterative solutions. On the other hand, a statistical approach, the so-called random coupling model (RCM) [9, 15, 16], has been developed to describe statistical properties of EM field in complex topologies such as circuits in boxes. The RCM enables one to predict the probability distribution function (PDF) of electric field on electronic components inside a partially shielded enclosure from a knowledge of wave power entering a "port" (such as a cooling vent) and the properties of the enclosure such as volume, frequency, and Q-factor. The predictions of the RCM have been validated in a number of experiments [16, 17].

We remark that the RCM-based stochastic approach is very powerful in predicting statistical descriptions of EM fields in simulation domains that are many wavelengths in size. On the other hand, for the solution of electrically small domains in complex electronic systems, the assumptions of RCM may not be valid. To address this challenge, we propose a hybrid method in which small electronic components (circuits, packages, PCBs, etc.) are modeled using first-principles and large portions (cavity enclosures) are modeled statistically. The **primary contributions** in this work are twofold: (**i**) a novel stochastic Green's function method for wave interaction with wave-chaotic media, which quantitatively describes the universal statistical property of chaotic systems through random matrix theory [11]; (**ii**) a hybrid deterministic and stochastic formulation based on an optimized multi-trace integral equation DD framework, which leads to a seamless integration of deterministic and stochastic solvers for the statistical characterization of in-situ IC electronics in short-wavelength wave-chaotic enclosures.

2. FORMULATION

This section provides an overview of the proposed work. We first introduce the methodology of domain decomposition for the hybrid formulation. A brief derivation of the stochastic Green's function in wave-chaotic media is presented next. Finally, the integration of deterministic and stochastic solvers and their application to statistical analysis of the in-situ IC performance and system behavior are discussed.

2.1. Domain Decomposition

We consider the solution of time-harmonic EM problem inside a large PEC enclosure, Ω, with its exterior boundary $\partial\Omega$. Two complex electronic components, defined by Ω_2 and Ω_3, are located inside the domain Ω. The region exterior to Ω_2 and Ω_3 is denoted by Ω_1, which is homogeneous and assumed to be free space, as illustrated in Fig. 2.

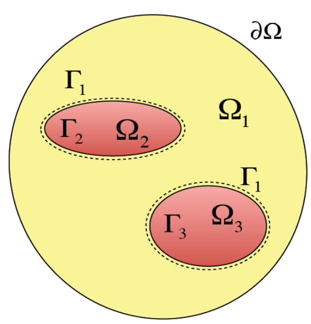


Figure 2: Notations for the hybrid DD method.

The first question to be answered in this subsection is how to obtain a suitable decomposed problem using the non-overlapping DD method. For simplicity, we consider a decomposition of computational domain into 3 sub-regions $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$, as shown in Fig. 2. The sub-region Ω_1 is modeled by boundary element (BE) solvers, and sub-regions Ω_2 and Ω_3 are solved by finite element (FE) solvers. The interface between boundary element and finite element sub-regions is denoted by Γ.

We employ the splitting idea and split the surface Γ into Γ_1 the surface seen from Ω_1, Γ_2 the surface seen from Ω_2, and Γ_3 the surface seen from Ω_3. The next step is to introduce two pairs of trace data on each sub-region interface. These traces are the Neumann trace $\mathbf{j}_1$, $\mathbf{j}_2$, $\mathbf{j}_3$ and Dirichlet trace $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, defined by:

$$\mathbf{j}_m = \frac{1}{\imath k_0}\hat{\mathbf{n}}_m \times \frac{1}{\mu_{rm}}\nabla \times \mathbf{E}_m; \qquad \mathbf{e}_m = \hat{\mathbf{n}}_m \times \mathbf{E}_m \times \hat{\mathbf{n}}_m \tag{1}$$

The introduced local multi-trace spaces allow a modular separation of interior FE and exterior BE sub-regions. Both FE and BE sub-regions are allowed to choose the discretization scheme, discrete formulation and solution strategy independently. Next, a general boundary value problem for the decomposed problem can be described as follows:

$$\mathcal{G}_m\left(\mathbf{e}_m, \mathbf{j}_m\right) = \mathbf{y}_m^{inc} \qquad in\ \Omega_m \tag{2}$$

$$\mathcal{B}_{mn}\left(\mathbf{e}_m, \mathbf{j}_m\right) = \mathcal{B}_{mn}\left(\mathbf{e}_n, -\mathbf{j}_n\right) \qquad on\ \Gamma_m \tag{3}$$

where $\mathcal{G}_m$ denotes the full-wave field solver for sub-region Ω_m. Equation (3) denotes the TC used to couple the Dirichlet and Neumann traces at the interfaces. $\mathcal{B}_{mn}$ usually consists of tangential pseudo-differential operators defined at the interface Γ_m. For instance, when the first (1st) order Robin-type TC [18–20] is employed, we have $\mathcal{B}_{mn}\left(\mathbf{e}, \mathbf{j}\right) := \mathbf{e} - \bar{\eta}_m\mathbf{j}$. In a recent work [21–23], we propose a second (2nd) order TC:

$$\mathcal{B}_{mn}\left(\mathbf{e}, \mathbf{j}\right) := \left(\mathcal{I} + \kappa_1\nabla_\tau \times \nabla_\tau \times + \kappa_2\nabla_\tau\nabla_\tau\cdot\right)\mathbf{e} - \left(\mathcal{I} + \kappa_3\nabla_\tau \times \nabla_\tau \times + \kappa_4\nabla_\tau\nabla_\tau\cdot\right)\bar{\eta}\mathbf{j} \tag{4}$$

at the interface between two different materials, where $\nabla_\tau \times \nabla_\tau \times$ and $\nabla_\tau\nabla_\tau\cdot$ are 2nd order tangential derivatives and τ denotes the tangential direction. κ_1, κ_2, κ_3, and κ_4 are the parameters that can be chosen to obtain rapidly converging algorithms.

We remark that the above mentioned non-overlapping DD methods have shown to be very effective in solving large multi-scale EM problems [23, 24]. To further improve the capability, boundary integral equation DD method [25] and Schwarz FE DD [14] can be employed for the solution of the BE and FE sub-regions, respectively. However, new challenges are encountered when those method are applied to analyze electronic systems in large complicated PEC enclosures.

In high-frequency regime, EM fields inside these enclosures show strong fluctuations and are very sensitive to system details. The study of such short-wavelength wave systems that exhibit chaotic ray dynamics is widely known as "wave chaos". In wave-chaotic systems, minor changes in the shape of the enclosure, or the reorientation of internal IC or electronics, can result in significantly different EM environments within the enclosure. Thus, deterministic solutions for a specific structure may be of limited practical value. This necessitates a quantitative statistical analysis of the in-situ IC performance and system behavior.

2.2. Stochastic Green's Function

It has been recognized that, despite the apparent complexity in wave-chaotic systems, they all possess certain universal statistical properties [15, 17]. Namely, the dynamics of the system are governed, in a qualitative way, by the symmetry of the system and not by the details of the interactions within the cavity. This motivates the derivation of stochastic Green's function, in which the statistical description depends only upon the value of a single dimensionless cavity loss-parameter.

To illustrate, we consider the Green function $G(\mathbf{r}, \mathbf{r}')$ for the scalar wave equation, $(\nabla^2 + k^2)G(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}')$. The first step is to expand the Green function in terms of the complete eigenfunctions of the system, i.e., $G(\mathbf{r}, \mathbf{r}') = \sum_j c_j\psi_j$, where $(\nabla^2 + k_j^2)\psi_j = 0$, $\int \psi_i(\mathbf{r})\psi_j(\mathbf{r})d\mathbf{r} = \delta_{ij}$ and $\sum_j \psi_j(\mathbf{r})\psi_j(\mathbf{r}) = \delta(\mathbf{r} - \mathbf{r}')$. Two theoretical tools are critical for this study: (**i**) eigenfunctions of the wave-chaotic media have statistical properties similar to those of a random superposition of many plane waves, $\psi_j(\mathbf{r}) \simeq \sum_{n=1}^{N} \alpha_n \cos(k_j\mathbf{e}_n \cdot \mathbf{r} + \beta_n)$, known as the random plane wave hypothesis [9]; (**ii**) eigenvalue spectra are statistically similar to the spectra of ensembles of random matrices, derived from Wigner's work on nuclear spectra [11]. Thus, we can obtain:

$$G(\mathbf{r}, \mathbf{r}') = \sum_j \frac{-\psi_j(\mathbf{r}')\psi_j(\mathbf{r})}{k^2 - k_j^2} \simeq \frac{-1}{\pi}\sum_j \frac{\Delta k^2\omega_j^2}{k^2 - k_j^2}\frac{J_0(k_j|\mathbf{r} - \mathbf{r}'|)}{4} \tag{5}$$

where ω_j is taken to be a Gaussian random variable with zero mean and unit variance, the eigenvalues k_j are distributed according to the random matrix hypotheses of Wigner. Moreover, we notice that the free space Green's function, $G_0(\mathbf{r}, \mathbf{r}')$, can be written as:

$$G_0(\mathbf{r}, \mathbf{r}') = \frac{i}{4} H_0^{(1)}(k|\mathbf{r} - \mathbf{r}'|) = \frac{i}{4} J_0(k|\mathbf{r} - \mathbf{r}'|) - \frac{\mathcal{P}}{\pi} \int_0^\infty \frac{k'^2 dk'}{k^2 - k'^2} \frac{J_0(k'|\mathbf{r} - \mathbf{r}'|)}{4} \tag{6}$$

Comparing (5) and (6), the sum over chaotic modes in (5) approximates on average the integral in (6) but with a statistically fluctuating contribution coming from modes with $k^2 - k_j^2$. We can rewrite the expression of Green's function in wave-chaotic media as:

$$G(\mathbf{r}, \mathbf{r}') = \mathrm{Re}[G_0(\mathbf{r}, \mathbf{r}')] + i\mathrm{Im}[G_0(\mathbf{r}, \mathbf{r}')]g \tag{7}$$

where g is a universal random variable that is described by random matrix theory [11]. A remarkable aspect of Eq. (7) is that it provides a clear separation between the universal statistical behavior of the wave-chaotic system denoted by g, and the deterministic coupling characteristics represented by $G_0(\mathbf{r}, \mathbf{r}')$.

2.3. Statistical Analysis

In the hybrid formulation, the electrically large sub-region, Ω_1, will be modeled by stochastic Maxwell solvers, and small sub-regions containing electronic components, Ω_2 and Ω_3, will be solved by deterministic Maxwell solvers. The interface between stochastic and deterministic sub-regions is denoted by Γ. The goal is to obtain the statistical prediction of voltages, currents, and EM fields inside electronic components of interested.

Specifically, for sub-region Ω_1, we write a multi-trace boundary integral equation formulation [21, 23] using the Dirichlet and Neumann traces, $\mathbf{e}_1$ and $\mathbf{j}_1$ defined on Γ_1, together with the stochastic Green's function investigated in (7). To gain further computational efficiency, the deterministic FE solver [13, 14] is employed to evaluate the numerical (discrete) Green's function, $\mathbf{Z}^-$, on the interface $\Gamma^- = \Gamma_1 \cup \Gamma_2$. Each column of $\mathbf{Z}^-$ corresponds to responses of electric and magnetic currents when excited by a unit source on the boundary surface Γ^-. As a result, it represents the discrete version of the Dirichlet-to-Neumann map in the deterministic sub-region. Once obtained, $\mathbf{Z}^-$ is integrated into the stochastic sub-region as exact transparent boundary conditions.

Finally, the solution of Dirichlet and Neumann traces on Γ_1 as a function of random variable g can be obtained by the Monte Carlo method or the Stochastic Collocation method with the collocation points determined by Clenshaw-Curtis nested quadrature rules [26]. To speed up the computation, we utilize the hierarchical interpolative decomposition [27, 28] to compress the dense BE matrix, and to speed up the coupling between stochastic sub-region boundaries. Once the statistical characterization of Dirichlet and Neumann traces is available, a backward postprocessing is employed to obtain the statistical prediction of EM fields inside Ω_2 and Ω_3.

3. NUMERICAL RESULTS

3.1. Deterministic Solution at Low-Mid Frequency

We consider a validation example by simulating two monopole antennas mounted inside a closed surface PEC cavity at 800 MHz–900 MHz. The computational domain is decomposed into three regions: 1) interior cavity region; 2) long monopole; and 3) short monopole, as shown in Fig. 3. The geometry of each monopole is also illustrated. After decomposition, the multi-trace boundary integral equation method [23] is used to discretize the cavity sub-domain Ω_1, and the finite element method is employed to discretize the antenna sub-domains Ω_2 and Ω_3. In the simulation, we excite the short monopole and use the long monopole as the receiving antenna. The computed S_{11} and S_{12} with respect to different operating frequencies are shown in Fig. 4. The measurement results conducted in Applied EM Group at University of New Mexico (UNM) are also given in Fig. 4. We observe a very good agreement between the results obtained by computation and measurement. We also notice that the computational results of S_{11} are slightly bigger than the measurement results. It might be due to the small loss introduced by those tiny slots on the edges and corners of cavity from imperfect fabrication.

3.2. Statistical Analysis at High Frequency

We proceed to study the case at a much higher frequency, 10 GHz. A X-band waveguide adapter is used to launch the EM fields within the cavity. At this frequency, the cavity is significantly overmoded and EM fields exhibit wave chaotic fluctuations.

To apply the proposed method, we first study the universal fluctuating quantity g in the stochastic Green's function. According to the RCM and random matrix theory, the only parameter

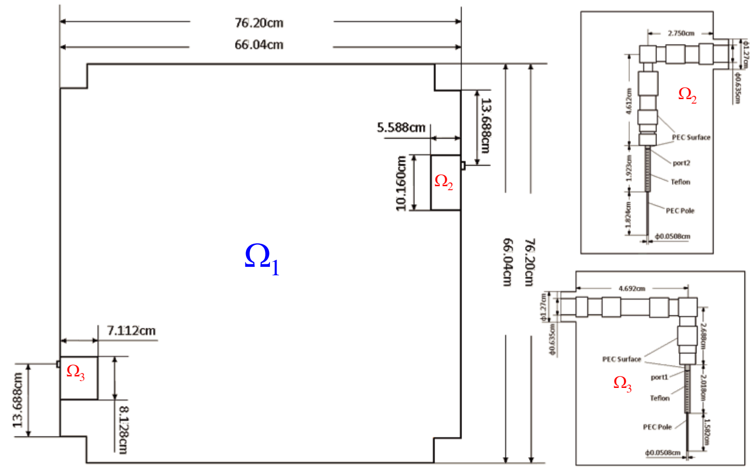


Figure 3: Configuration of the validation example and geometry of the antennas.

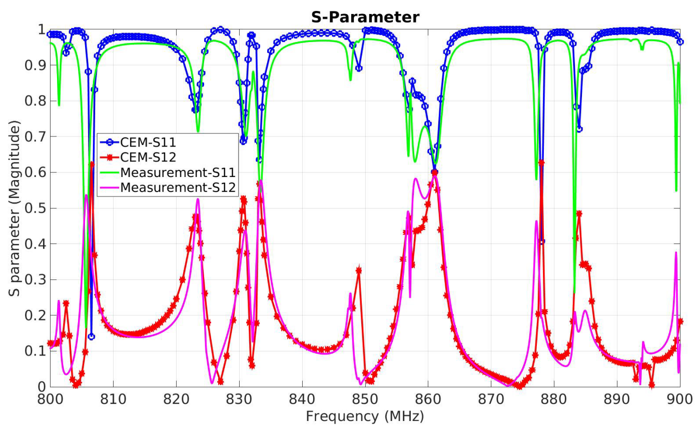
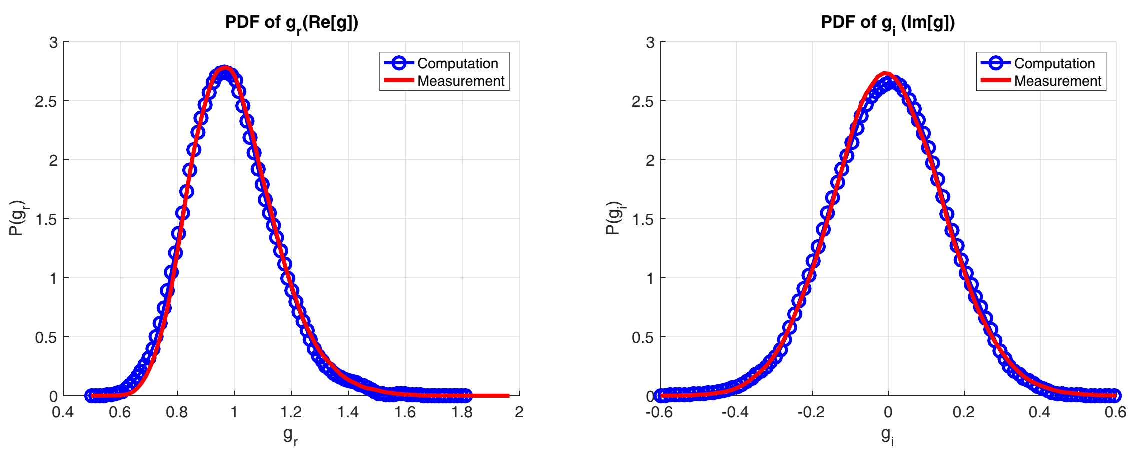


Figure 4: Comparison of S-parameters obtained by computation and measurement.



(a) Real part

(b) Imaginary part

Figure 5: PDF of the universal random variable g.

that determines the statistics of g is the dimensionless cavity loss-parameter $\alpha = \frac{k^2}{\Delta k_n^2 Q}$, where Δk_n^2 is the mean-spacing of the adjacent eigenvalues of the Helmholtz operator predicted by Weyl Formula [6] and Q represents the loaded quality-factor of the cavity. Shown in Fig. 5 is the probability distribution function (PDF) of real and imaginary part of g obtained by computational predictions and measurement results.

Next, we apply the proposed hybrid formulation to analyze the PDF of S_{11} of the waveguide launching adapter in a frequency range 10 GHz–10.1 GHz. In the experimental setup, 1601 frequency sample points are chosen. At each frequency point, the internal mode stirrer is rotated through 200 positions over 360 degrees. The resulting S_{11} and its ensemble average are shown in Fig. 6. Finally, the PDF of S_{11} obtained by experimental results and computational prediction using the proposed method are given in Fig. 7. We observe a very good agreement in the patterns of PDF plots. There is a slight left shift in experimental results due to the small loss in the cavity we discussed earlier.

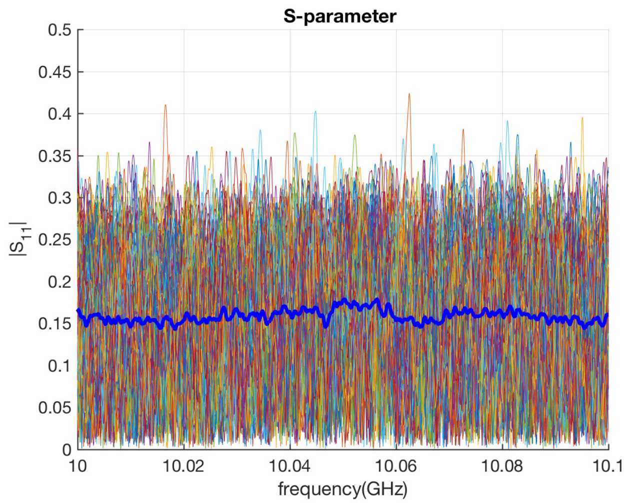


Figure 6: Experimental results of S_{11} for a sequence of mode stirrer locations and their ensemble average (curve in blue color).

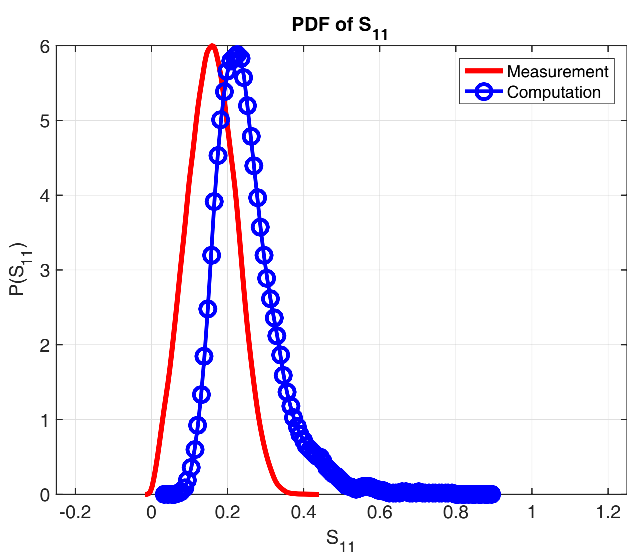


Figure 7: PDF of the S_{11} obtained by measurement data and computational prediction.

4. CONCLUSION

The proposed work aims to answer a fundamental challenge in the quantitative statistical modeling of IC and electronics in large complex enclosure. A new stochastic dyadic Green's function method is introduced for wave interaction with wave-chaotic media, which quantitatively describes universal statistical properties of chaotic systems through random matrix theory. It also leads to a seamless integration of deterministic and stochastic formulation for the statistical characterization of in-situ IC electronics in short-wavelength wave-chaotic enclosures.

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