Statistical Characterization of Complex Enclosures with Distributed Ports

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***Abstract*—A statistical model, the Random Coupling Model, that describes the coupling of radiation into and out of large electrical enclosures is described and generalized. Particular attention is paid to the case in which the ports are electrically large and described by multiple modes (distributed ports). We find a compact expression for a model of the enclosure impedance that can be used to generate probability distributions for fields at the enclosures ports. Results are of interest in the evaluation of power leakage in complex metallic structures and reverberation chambers, and the evaluation of the effectiveness of shielding in the presence of apertures.**

I. Introduction

A statistical approach is often taken to describe the excitation of, and field distribution in, complicated electromagnetic enclosures [1], [2], [3], [4], [5], [6], [7]. The basic idea is that due to a combination of the complexity of the geometry, uncertainties in precise locations of boundaries or objects in the enclosure, and the sensitive dependence of the fields to the frequency of excitation, first principles solutions of the Maxwell equations are not practical, or needed, to make predictions of distributions of quantities of interest. In the statistical approach one instead attempts to predict the probability distribution for measuring a particular field value using a model based on a minimum of information about the enclosure of interest. Such a model is relevant to prediction and design of immunity methods in complex electromagnetic environments.

One such statistical approach is the Random Coupling Model (RCM) [6], [7]. RCM is based on the use of Random Matrix Theory (RMT) [8] that has found wide application in mesoscopic and nuclear physics [8], [9], [10], [11], [12]. The model incorporates system specific information about the near field behaviour of the ports of the enclosure, the volume of the enclosure, and a measure of the enclosures quality factor, and predicts the statistical behaviour of the enclosures scattering parameters (actually the impedance matrix from which the scattering matrix can be obtained). This model was introduced in [5], [6], [7] for a specific geometry, namely planar cavities excited by point like sources. The validity of the model has been verified by comparison with both numerically calculated and experimentally measured scattering parameters.

The formulation of the model for the case of three-dimensional cavities, with arbitrary polarization of the fields inside, and with ports that are large compared with a wavelength has not previously been presented. The purpose of this paper is to make such a presentation.

II. The Random Coupling Model

A. Definition of RCM

RCM works by first identifying a suitable set of voltages and currents that are linearly related and that can be used to describe the interaction of the fields within the cavity with signals to and from the outside world. In the case of planar cavities in [5], [6], [7], voltages and currents were those at the ends of transmission lines that were used to couple to the enclosure. Then RCM, provides a model for the linear relation between these voltages and currents that mimics the behaviour of the fields in the enclosure. The model is based on the following approach. First one imagines representing the fields inside the enclosure in a complete basis of modes, and calculates the excitation of these modes due to coupling to the ports. One then writes a formal expression for the matrix impedance that involves the modes and their resonant frequencies. The real modes and resonant frequencies are too complicated, and too sensitive to details, to calculate. So, these are replaced by representations that are based on RMT and the assumption that modes appear to be random superpositions of plane waves. The result is a compact expression for a model of the cavity impedance matrix

$$\underline{\underline{Z}}^{cav} = i\Im\left\{\underline{\underline{Z}}^{rad}\right\} + \left[\underline{\underline{R}}^{rad}\right]^{1/2} \cdot \underline{\underline{\xi}} \cdot \left[\underline{\underline{R}}^{rad}\right]^{1/2} \quad , \qquad (1)$$

where $\underline{\underline{Z}}^{rad} = \underline{\underline{R}}^{rad} + i\Im\left(\underline{\underline{Z}}^{rad}\right)$ is in the simplest theory an $N_p \times N_p$ diagonal matrix (where N_p is the number of ports) whose elements are the complex radiation impedances of the ports, and we have adopted the notation that a double underline indicates a matrix quantity. Here, the radiation impedance provides the linear relation between voltages and currents at a port in the case in which waves are allowed to enter the enclosure through the port but not return, as if they were absorbed in the enclosure. The matrix $\underline{\underline{\xi}}$ is an element of the Lorentzian ensemble [8] and can be defined for a lossless cavity as

$$\underline{\underline{\xi}} = \frac{i}{\pi} \sum_n \frac{\Delta k^2 \underline{w}_n \underline{\tilde{w}}_n}{(k_0^2 - k_n^2)} \ . \qquad (2)$$

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Here $\underline{w}_n$ is a vector of uncorrelated, zero mean, unit width Gaussian random variables, and k_{n2} are the eigenvalues of a matrix selected from the Guassian Orthogonal Ensemble (GOE) [13], where the central eigenvalue is shifted to be close to $k_0^2 = \omega^2/c^2$ and ω, is the frequency of excitation. The shift implies that $\underline{\underline{\xi}}$ has zero mean. The eigenvalues are scaled so that the average spacing between eigenvalues near the central one is Δk^2, which is selected to match the mean spacing of resonances of the enclosure in the frequency range of interest. The effect of internal losses, or additional ports (beyond the N_p already considered), can be treated simply by making the replacement $k_0^2 \rightarrow k_0^2 (1 + j/Q)$ in the denominator of (2). In this way the statistics of the $\underline{\underline{\xi}}$ matrix depend on the loss parameter $\alpha = k_0^2/\left(Q\Delta k^2\right)$. It is through the matrix $\underline{\underline{\xi}}$ that the propagation of waves in the enclosure from one port to another and back is modeled.

We now discuss the ways in which the impedance (or admittance) matrix is defined for a port, and then discuss how its values are determined. We generally identify three situations of interest, which we label the *terminal* case, the *closed aperture* case, and the *open aperture* case. The precise definition of the impedance matrix will vary in these cases, as will the method of calculation of the matrix. However, all three of these cases can still be treated within the RCM. The terminal case applies to the situation where a port is excited through a single mode transmission line, and the excitation of the port can be prescribed by a single variable: the voltage, or current, or amplitude of the incident wave on the transmission line. Our studies [5], [6], [7] of the excitation of cavities by signals on cables are examples of this case. In addition, a terminal or lead on an integrated circuit can be treated as an example of this case if one considers the input to the circuit as a lumped element and the conductors and dielectric material surrounding the integrated circuit as an antenna. In the terminal case, determination of the radiation impedance becomes equivalent to solving for the fields surrounding an antenna that is driven by a transmission line. It is thus important to account for the geometry and dielectric properties of the material surrounding, within several wavelengths, the terminal. Calculation of the port impedance can be quite complicated as it involves the self-consistent determination of the current in all conductors and polarization of all dielectrics near the port. A simple case is that of an antenna that is small compared with a wavelength. In this case the current distribution in the antenna is fixed. An example of this is that of a coaxial antenna in a two dimensional cavity [5], [6], [7].

The procedure for treating a pin on an integrated circuit in the terminal case is as follows. One imagines that there is a fixed current source within the integrated circuit that excites the conductors surrounding the integrated circuit and that radiates energy away form the integrated circuit. The voltage that appears at the terminal, divided by the fixed current defines the radiation impedance for that port. When analyzing the statistics of the voltages that appear at that terminal, when the integrated circuit is placed in the cavity, it is necessary to account for the impedance seen looking into the integrated circuit. In the Random Coupling Model one assumes that the port terminal is connected to a load with this impedance.

B. Superposition of current distributions

One case where a closed-form expression can be obtained is the calculation of the impedance for a set of ports that can be modeled as a superposition of fixed current distributions. We assume the current density profile can be written as the product of a port current, I_p , and a basis of spatially dependent profile functions, $\mathbf{u}_p(\mathbf{x})$

$$\mathbf{J}(\mathbf{x}) = \sum_p \mathbf{u}_p(\mathbf{x}) I_p, \ , \tag{3}$$

here the sum is over elements of the basis. The corresponding port voltage is then defined as

$$V_p = -\int d^3x \, \mathbf{u}_p(\mathbf{x}) \cdot \mathbf{E}(\mathbf{x}) \ . \tag{4}$$

This definition preserves power balance,

$$P = \frac{1}{2}\Re\left\{\int d^3\mathbf{E}\cdot\mathbf{J}^*\right\} = \frac{1}{2}\Re\left\{\sum_p V_p I_p^*\right\} \ . \tag{5}$$

To calculate the radiation impedance we insert expression (3) in the Maxwell equations, Fourier transform in space, solve for the Fourier transform of the electric field, take the inverse transform, and insert the expression for the electric field in (4). The result is

$$V_p = \sum_{p'} Z_{pp'}^{rad}(k_0) I_{p'} \ , \tag{6}$$

where

$$Z_{pp'}^{rad}(k_0 = \omega/c) = \sqrt{\frac{\mu}{\epsilon}} \int \frac{d^3k}{(2\pi)^3} \frac{ik_0}{k_0^2 - k^2} \tilde{\mathbf{u}}_p \cdot \underline{\underline{\Delta}}_1 \cdot \tilde{\mathbf{u}}_{p'}^* \ . \tag{7}$$

Here

$$\underline{\underline{\Delta}}_1 = \frac{\mathbf{1}k^2 - \mathbf{kk}}{k^2} + \frac{\mathbf{kk}}{k^2 k_0^2}\left(k_0^2 - k^2\right) \ . \tag{8}$$

The radiation impedance matrix is a complex quantity. The residue at the pole $k = k_0$ in (7) gives the radiation resistance

$$R_{pp'}^{rad}(k_0) = \Re\left(Z_{pp'}^{rad}\right) = \sqrt{\frac{\mu}{\epsilon}} \int \frac{k_0^2 d\Omega_k}{16\pi^2} \tilde{\mathbf{u}}_p^* \cdot \frac{\mathbf{1}k^2 - \mathbf{kk}}{k^2} \cdot \tilde{\mathbf{u}}_{p'}^* \ , \tag{9}$$

where Ω_k is the two dimensional solid angle of the wave vector $\mathbf{k}$. The radiation resistance is frequency dependent through $k_0 = \omega/c$, and we note that there is an implicit k_0 dependence through the Fourier transforms of the port functions. The impedance of a cavity excited by the same port currents can be expressed by expanding the fields inside the cavity in a basis of electric and magnetic modes

$$\mathbf{E} = \sum_n \left(V_n^{em}\mathbf{e}_n^{em}(\mathbf{x}) + V_n^{es}\mathbf{e}_n^{es}(\mathbf{x})\right) \ , \tag{10}$$

and

$$\mathbf{H} = \sum_n I_n^{em}\mathbf{h}_n^{em}(\mathbf{x}) \ . \tag{11}$$

Here the electromagnetic modes satisfy the pair of equations, $-ik_n\mathbf{e}_n^{em} = \nabla\times\mathbf{h}_n^{em}$, and $ik_n\mathbf{h}_n^{em} = \nabla\times\mathbf{e}_n^{em}$ with the tangential components of the electric field equal to zero on the boundary.

The electrostatic modes are irrotational $\mathbf{e}_n^{es} = -\nabla\phi_n$, where the potential satisfies the Helmoltz equation $\left(\nabla^2 + k_n^2\right)\phi_n = 0$, with the Dirichelet boundary conditions, $\phi_n|_B = 0$. The electrostatic modes are needed to represent electric fields that have non-vanishing divergence inside the cavity. It can be shown that all the electric field modes are orthogonal. The mode amplitudes are determined by projecting the Maxwell equations onto the basis functions for each field type. The result of this action is an expression for the port voltages that is equivalent to (6) except that the radiation impedance matrix is replaced by a cavity impedance matrix

$$Z_{pp'}^{cav}(k_0) = \sqrt{\frac{\mu}{\epsilon}}\sum_n\left(\frac{ik_0}{k_0^2-k_n^2}\frac{\mathbf{u}_{pn}^{em}\mathbf{u}_{p'n}^{em}}{V^{em}} + \frac{i}{k_0}\frac{\mathbf{u}_{pn}^{es}\mathbf{u}_{p'n}^{es}}{V^{es}}\right), \tag{12}$$

where $\mathbf{u}_{pn}^{(\cdot)} = \int d^3x\,\mathbf{e}_n^{(\cdot)}(\mathbf{x})\cdot\mathbf{u}_p(\mathbf{x})$, is the projection of the magnetic field of the cavity mode onto the port current profile, and $V^{(\cdot)} = \int d^3x\left|\mathbf{e}_n^{(\cdot)}\right|^2$ is a normalization factor for the eigenfunctions. Expression (12) is general and gives an exact expression for the impedance matrix of a lossless cavity in terms of the cavity modes. As mentioned before, RCM proposes modifying (12) in the following way:

- replace the exact eigenmodes with putative modes corresponding to random superpositions of plane waves;
- replace the exact spectrum k_n^2 of eigenmodes with one generated by a random matrix from the GOE.

Carrying out the first step, we write for the components of the electromagnetic eigenmodes,

$$\mathbf{e}_n^{(em)} = \lim_{N\to\infty}\sqrt{\frac{2}{N}}\sum_{j=1}^{N}\Re\left[\mathbf{b}_j\,\exp\left(\theta_j + \mathbf{k}_j\cdot\mathbf{x}\right)\right], \tag{13}$$

where θ_j are uniformly distributed in the interval $[0, 2\pi]$, $|\mathbf{k}_j| = k_n$, with the direction of $\mathbf{k}_j$ uniformly distributed over a 4π solid angle, $|\mathbf{b}_j = 1|$, with $\mathbf{b}_j$ uniformly distributed in angle in the plane perpendicular to $\mathbf{k}_j$. Except as mentioned, all random variables characterizing each plane wave are independent. A similar expression can be made for the scalar potential ϕ_n generating the electrostatic modes. With eigenfunctions expressed as a superposition of random plane waves, each factor $\mathbf{u}_{pn}^{(\cdot)}$ appearing in (12) becomes a zero mean Guassian random variable. The correlation matrix between two such factors can then be evaluated by forming the product of two terms, averaging over the random variables parameterizing the eigenfunctions and taking the limit $N\to\infty$. We find for the electromagnetic modes the following expectation value,

$$\left\langle\frac{\mathbf{u}_{pn}^{em}\mathbf{u}_{p'n}^{em}}{V^{em}}\right\rangle = \int\frac{d\Omega_k}{8\pi V}\tilde{\mathbf{u}}_p\cdot\left[\frac{\mathbf{1}k^2-\mathbf{kk}}{k^2}\right]\cdot\tilde{\mathbf{u}}_{p'}^* . \tag{14}$$

Here, $|\mathbf{k}| = k_n$, Ω_k represents the spherical solid angle of $\mathbf{k}$, and V is the volume of the cavity. A similar analysis of the electrostatic modes gives

$$\left\langle\frac{\mathbf{u}_{pn}^{es}\mathbf{u}_{p'n}^{es}}{V^{es}}\right\rangle = \int\frac{d\Omega_k}{4\pi V}\tilde{\mathbf{u}}_p\cdot\left[\frac{\mathbf{kk}}{k^2}\right]\cdot\tilde{\mathbf{u}}_{p'}^* . \tag{15}$$

The connection between the cavity case (12) and the radiation case (7) is now beginning to emerge. Specifically, we note that the factors $\mathbf{u}_{pn}^{em}$ are zero mean Gaussian random variables with a correlation matrix given by (15). We can express the product $\mathbf{u}_{pn}^{em}\mathbf{u}_{p'n}^{em}$ in terms of uncorrelated zero mean, unit width Gaussian random variables by diagonalizing the correlation matrix. We now introduce matrix notation and represent the pp' element of the product as

$$\frac{\mathbf{u}_{pn}^{em}\mathbf{u}_{p'n}^{em}}{V^{em}} = 2\sqrt{\frac{\epsilon}{\mu}}\Delta k_n\left\{\left[\underline{\underline{R}}^{rad}\right]^{1/2}\cdot\underline{w}_n\underline{w}_n^T\cdot\left[\underline{\underline{R}}^{rad}\right]^{1/2}\right\}_{pp'}, \tag{16}$$

where $\underline{\underline{R}}^{rad}$ is the radiation resistance matrix (9), $\underline{w}_n$ is a vector of uncorrelated, zero mean, unit width Gaussian random variables, and $\Delta k_n = \pi^2/\left(Vk_n^2\right)$ is the mean spacing between electromagnetic eigenmodes of a three dimensional cavity. Substituting (16) into (12) we have

$$Z_{pp'}^{cav} = \left\{\sum_n\frac{2ik_0\Delta k_n}{\pi\left(k_0^2-k_n^2\right)}\left[\underline{\underline{R}}^{rad}\right]^{1/2}\cdot\underline{w}_n\underline{w}_n^T\cdot\left[\underline{\underline{R}}^{rad}\right]^{1/2}\right\}_{pp'} + i\,X_{pp'}^{es}(k_0) . \tag{17}$$

Here, in the limit of a large cavity, we have approximated the sum of the pairs of gaussian random variables representing the electrostatic contribution to the cavity impedance by their average values and using the mean spacing formula for electrostatic modes converted the sum to an integral $\sum_n\Delta k_n\to\int_0^\infty dk_n$. Similarly we can evaluate the expected value of an element of the cavity impedance matrix by averaging over the $\underline{w}_n$, and replacing the sum over eigenvalues in (17) with a continuous integral. Comparing with the definitions (6) and (7) we see

$$\left\langle Z_{pp'}\right\rangle = iX_{pp'}^{rad}(k_0) = i\Im\left\{Z_{pp'}^{rad}\right\} . \tag{18}$$

The second step is to replace the exact spectrum of eigenvalues, k_n^2 in (17) by a spectrum produced by random matrix theory. These are adjusted so that the mean spacing matches that of the actual cavity in the vicinity $k_n\approx k_0$. The result is then expressed in compact form as (1). The radiation impedance defined by (6) and (7) has off-diagonal components that describe the fields induced at one port as a result of currents flowing in another. If the ports are close together the off-diagonal terms are comparable to the diagonal ones, and they have a significant effect on the predicted cavity impedance. However, these effects are captured by (1) simply by using the non-diagonal radiation impedance. If the ports are separated by a distance of many wavelengths these off diagonal terms are smaller than the diagonal terms and have a less significant effect on the cavity impedance. Further, in the case of well-separated ports the radiation impedance matrix includes only the effect of direct propagation from one port to another, and not the effect of ray paths that bounce from a wall of the cavity in propagating from one port to another. We have recently shown [15], [16] how these ray paths can be included in the development of a model cavity impedance, but we will not pursue this effect further here. (Generally, these orbits do not contribute coherently when one considers ensemble averages over large enough ranges of frequency or large enough variations in cavity shape.)

C. Planar apertures

Expressions (7) and (17) apply when the port is represented as a current distribution with a number of components that have different spatial profiles. Another case of interest is that of an enclosure excited through an aperture. In this case the port is characterized by an impedance (or scattering) matrix that has a dimension equal to the number of modes used to represent the fields in the aperture. For example, suppose the port is treated as an aperture in a planar conductor whose surface normal $\hat{n}$ is parallel to the z-axis. The components of the fields transverse to z in the aperture can be expressed as a superposition of a basis of modes (for example, the modes of a waveguide with the same cross sectional shape as the aperture)

$$\mathbf{E}_t = \sum_s V_s \mathbf{e}_s\left(\mathbf{x}_\perp\right) \ , \tag{19}$$

and

$$\mathbf{H}_t = \sum_s I_s \hat{n} \times \mathbf{e}_s\left(\mathbf{x}_\perp\right) \ , \tag{20}$$

where $\mathbf{e}_s$ is the basis mode, (having only transverse fields) normalized such that $\int_{aperture} dx_\perp^2 \, \left|\mathbf{e}_s\right|^2 = 1$ and $\hat{n}$ is the outward normal to the cavity, which we take to be in the z-direction. In the radiation case we solve the Maxwell equations in the half space $z > 0$ subject to the boundary conditions that $\mathbf{E}_t = 0$ on the conducting plane except at the aperture where it is given by (19). We then evaluate the transverse components of the magnetic field on the plane $z = 0$, and project them on to the basis $\hat{n} \times \mathbf{e}_s\left(\mathbf{x}_\perp\right)$ at the aperture to find the magnetic field amplitudes I_s in (20). The result is a matrix relation between the magnetic field amplitudes and the electric field amplitudes in the aperture

$$I_s = \sum_{s'} Y_{ss'}^{rad}\left(k_0\right) V_{s'} \ , \tag{21}$$

where

$$Y_{ss'}^{rad}\left(k_0 = \omega/c\right) = \sqrt{\frac{\mu}{\epsilon}} \int \frac{d^3k}{\left(2\pi\right)^3} \frac{2ik_0}{k_0^2 - k^2} \tilde{\mathbf{e}}_s \cdot \underline{\underline{\Delta}}_2 \cdot \tilde{\mathbf{e}}_{s'}^* \ , \tag{22}$$

with

$$\underline{\underline{\Delta}}_2 = \frac{\mathbf{k}_\perp \mathbf{k}_\perp}{k_\perp^2} + \left(\frac{k^2 - k_\perp^2}{k^2} + \frac{\left(k_0^2 - k^2\right) k_\perp^2}{k^2 k_0^2}\right) \frac{\left(\mathbf{k} \times \hat{n}\right)\left(\mathbf{k} \times \hat{n}\right)}{k_\perp^2} \times \left(k_0^2 - k^2\right) \ . \tag{23}$$

and

$$\tilde{\mathbf{e}}_s = \int_{aperture} d^2 x_\perp \, \exp\left(-i\mathbf{k}_\perp \cdot \mathbf{x}_\perp\right) \mathbf{e}_s\left(\mathbf{x}_\perp\right) \ . \tag{24}$$

If we repeat the process, but assume the aperture opens into a cavity rather than an infinite half space, the radiation admittance is replaced by a cavity admittance. Following the steps leading to (17), i.e., under the assumptions that the eigenmodes of the closed cavity can be replaced by superpositions of random plane waves, and the spectrum of the cavity eigenmodes can be replaced by one corresponding to a random matrix from the Gaussian Orthogonal Ensemble, the statistical properties of the cavity admittance can be represented as follows

$$\underline{\underline{Y}}^{cav} = i\Im\left\{\underline{\underline{Y}}^{rad}\right\} + \left[\underline{\underline{G}}^{rad}\right]^{1/2} \cdot \underline{\underline{\xi}} \cdot \left[\underline{\underline{G}}^{rad}\right]^{1/2} = i\Im\left\{\underline{\underline{Y}}^{rad}\right\} + \delta\underline{\underline{Y}}^{cav} \ , \tag{25}$$

where $\underline{\underline{G}}^{rad}$ is the radiation conductance, i.e., $\underline{\underline{Y}}^{cav} = \underline{\underline{G}}^{rad} + i\Im\left\{\underline{\underline{Y}}^{rad}\right\}$, the matrix $\underline{\underline{\xi}}$ is the same as defined in (2), and $\delta\underline{\underline{Y}}^{cav}$ is the fluctuating part of $\underline{\underline{Y}}^{cav}$.

Expression (21) is derived for an arbitrary basis. If the aperture is connected to a waveguide ($z < 0$) an appropriate basis is the set of modes of a waveguide. A subset of the modes will propagate in the waveguide, and the others will be cut-off. For a sufficiently long waveguide the propagating modes represent the channels for energy to enter and leave the cavity. The cut-off modes will be present in the waveguide, but localized to the region near $z = 0$. To account for the cut-off modes we imagine partitioning the voltages and currents in (19) and (20) into the two groups: propagating and cut-off modes. For the cut-off modes we insert $I_s = Y_{0,s} V_s$ in (20) where $Y_{0,s}$ is the imaginary admittance associated with cut-off mode s in the waveguide. We can then imagine solving for the voltages of the cut-off modes in terms of the voltages of the propagating modes. Substituting the cut-off voltages back into (21) gives a proper admittance matrix relation involving only the propagating modes. We assert that relation (25) applies to the statistics of the reduced dimension cavity matrix as well. The elements of the reduced radiation admittance matrix will be different from the corresponding elements of the full radiation admittance matrix due to the effect of the cut-off modes. However, the relation between the radiation admittance matrix and the cavity admittance matrix is preserved.

We now consider the open aperture case where the aperture is illuminated by a plane wave incident with a wave vector $\mathbf{k}^{inc}$ and polarization of magnetic field $\mathbf{H}^{inc}$ that is perpendicular to $\mathbf{k}^{inc}$. We imagine writing the fields for $z < 0$ as the sum of the incident wave, the wave that would be specularly reflected from an infinite planar surface, and a set of outgoing waves associated with the presence of the aperture. The incident and specularly reflected waves combine to produce zero tangential electric field on the plane $z = 0$. Thus, the outgoing waves for $z < 0$ associated with the aperture can be expressed in terms of the electric fields in the aperture just as the outgoing waves for $z > 0$ can, the two cases being mirror images. So, relation (19) continues to represent the tangential electric fields in the plane $z = 0$. For the magnetic field we have separate expansions for $z > 0$ and $z < 0$. The electric field amplitudes are then determined by the condition that the magnetic fields are continuous in the aperture at $z=0$. For $z=0^+$ we have $\mathbf{H}_t^> - \sum_s I_s^> \hat{n} \times \mathbf{e}_s(\mathbf{x}_\perp)$, with $\underline{I}^> = \underline{\underline{Y}}^> \cdot \underline{V}$, where $\underline{\underline{Y}}^>$ is either the radiation admittance matrix or the cavity admittance matrix depending on the circumstance. For $z=0^-$ we have $\mathbf{H}_t^< - \sum_s I_s^< \hat{n} \times \mathbf{e}_s(\mathbf{x}_\perp) + 2\mathbf{H}^{inc} \exp\left(i\mathbf{k}^{inc} \cdot \mathbf{x}_\perp\right)$, where $\underline{I}^< = \underline{\underline{Y}}^{rad} \cdot \underline{V}$ (the minus sign accounts for the mirror symmetry) and the factor of two multiplying the incident field comes from the addition of the incident and specularly reflected magnetic fields. Projecting the two magnetic field expressions on the aperture basis, and equating the amplitudes gives

$$\left(\underline{\underline{Y}}^> + \underline{\underline{Y}}^{rad}\right) \cdot \underline{V} = 2\underline{I}^{inc} \; , \tag{26}$$

where $I_s^{inc} = -\hat{n} \cdot \tilde{\mathbf{e}}_s\left(\mathbf{k}_\perp^{inc}\right) \times \mathbf{H}^{inc}$, and $\tilde{\mathbf{e}}_s$ is the Fourier transform of the aperture electric field. Expression (26) can be inverted to find vector of voltages, and then from (21) the power passing through the aperture can be determined.

The terminal case leads naturally to consideration of an impedance matrix while the aperture case leads to consideration of an admittance matrix. In cases involving both apertures and terminals it is possible to consider a hybrid matrix. In that case we construct an input column vector $\underline{\phi}$ that consists of the aperture voltages and terminal currents and an output vector $\underline{\psi}$ that consists of the aperture currents and terminal voltages

$$\underline{\phi} = \begin{pmatrix} \underline{V}^A \\ \underline{I}^P \end{pmatrix} \; , \tag{27}$$

and

$$\underline{\psi} = \begin{pmatrix} \underline{I}^A \\ \underline{V}^P \end{pmatrix} \; , \tag{28}$$

where $\underline{V}^{A,P}$ are the aperture and port voltages, and $\underline{I}^{P,A}$ are the aperture and port currents. These are then related by a hybrid matrix $\underline{\underline{T}}$, $\underline{\psi} = \underline{\underline{T}} \cdot \underline{\phi}$, where

$$\underline{\underline{T}} = i\Im\left\{\underline{\underline{U}}\right\} + \left[\underline{\underline{V}}\right]^{1/2} \cdot \underline{\underline{\xi}} \cdot \left[\underline{\underline{V}}\right]^{1/2} \; . \tag{29}$$

Here, the matrices $\underline{\underline{U}}$ and $\underline{\underline{V}}$ are block diagonal

$$\underline{\underline{U}} = \begin{bmatrix} \underline{\underline{Y}}^{rad} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{Z}}^{rad} \end{bmatrix} \; , \tag{30}$$

and $\underline{\underline{V}} = \Re\left\{\underline{\underline{U}}\right\}$. The dimension of $\underline{\underline{U}}$ and $\underline{\underline{V}}$ is $(N_s + N_p) \times (N_s + N_p)$, where N_p is the number of port currents and N_s is the number of port voltages. Here, we have assumed that the ports and apertures are sufficiently separated such that the off diagonal terms in $\underline{\underline{U}}$, describing the excitation of port voltages by aperture voltages, and aperture currents by port currents, are approximately zero. In this case we can take the square root in (29)

$$\left[\underline{\underline{V}}\right]^{1/2} = \begin{bmatrix} \left[\underline{\underline{Y}}^{rad}\right]^{1/2} & \underline{\underline{0}} \\ \underline{\underline{0}} & \left[\underline{\underline{Z}}^{rad}\right]^{1/2} \end{bmatrix} \; . \tag{31}$$

Matrix $\underline{\underline{T}}$ defined in (29) can then be used to calculate the signal received at ports due to the illumination of an aperture. Specifically, let us assume that the aperture is illuminated by a plane-wave as in (26) so that $\underline{I}^{inc}$ is known. The aperture currents then must be given by $\underline{I}^A = -\underline{\underline{Y}} \cdot \underline{V}^A + 2\underline{I}^{inc}$.

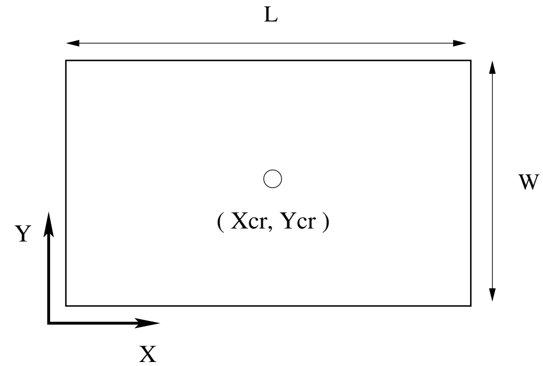

Fig. 1. Planar rectangular aperture adopted for preliminary computations based on RCM.

Let us also assume that the ports are connected to load impedances described by the diagonal matrix $\underline{\underline{Z}}^L$. The elements of $\underline{\underline{Z}}^L$represent the characteristic impedance of ports fed by transmission lines or the internal impedance of circuit elements that are being treated as ports. The vector of aperture voltages and port currents is then given by the matrix equation

$$\begin{aligned} \begin{pmatrix} \underline{\underline{Y}}^{rad} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{Z}}^L \end{pmatrix} \cdot \begin{pmatrix} \underline{V}^A \\ \underline{I}^P \end{pmatrix} + \underline{\underline{T}} \cdot \begin{pmatrix} \underline{V}^A \\ \underline{I}^P \end{pmatrix} \\ = \begin{pmatrix} 2\underline{I}^{inc} \\ \underline{0} \end{pmatrix} \end{aligned} \tag{32}$$

Once (32) is solved for the port currents, the port voltages, including the voltages on designated terminals, is given by $\underline{V}^P = -\underline{\underline{Z}}^L \cdot \underline{I}^P$.

III. MONTECARLO SIMULATION OF CHAOTIC CAVITIES: RECTANGULAR APERTURE

The effectiveness of the present approach is now illustrated by performing Montecarlo evaluations of expression (25). In particular, we focus on the computation of its fluctuating part $\delta\underline{\underline{Y}}^{cav}$ given in terms of aperture conductance, which depends on k_0, aperture geometry, and related field distribution, through

$$\begin{aligned} G_{ss'}^{rad}\left(k_0 = \omega/c\right) = \sqrt{\frac{\mu}{\epsilon}} \int \frac{k_0^2 d\Omega_k}{8\pi^2} \tilde{\mathbf{e}}_s \\ \cdot \left[\frac{\mathbf{k}_\perp \mathbf{k}_\perp}{k_\perp^2} + \frac{(\mathbf{k} \times \hat{n})(\mathbf{k} \times \hat{n})}{k_\perp^2}\right] \cdot \tilde{\mathbf{e}}_{s'}^* \; . \end{aligned} \tag{33}$$

The computation of smoothly varying $\Im\left(\underline{Y}_{ss'}^{rad}\right)$ involves the Cauchy principal value calculation and is deferred for the time being. The radiation conductance $\underline{G}_{ss'}^{rad}$ is computed assuming a proper set of orthogonal basis functions, $\mathbf{e}_s$ for transverse field distribution of the aperture and then Fourier transforming with respect to transverse coordinates. We adopted the canonical shape of [17, Eq. (2)], valid for both electrically narrow and large planar rectangular apertures of Fig. 2 [18]. The fluctuating matrix $\underline{\underline{\xi}}$ represents the chaotic scattering taking place within the cavity. It depends only on the loss parameter α [6], [7], and will be computed by Montecarlo simulation.

To do this we used the strategy of [5] for generating $\underline{\underline{\xi}}$,

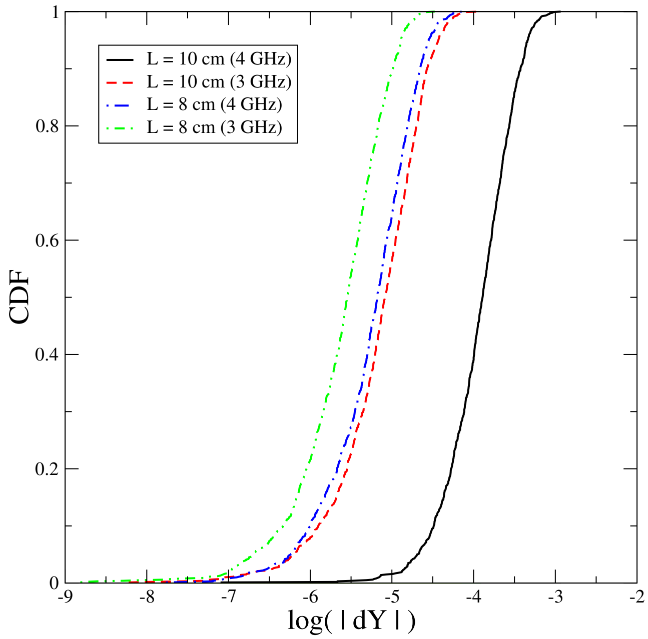

Fig. 2. CDF of the fluctuating part of the cavity admittance: $s = 0$, $s^{'} = 0$.

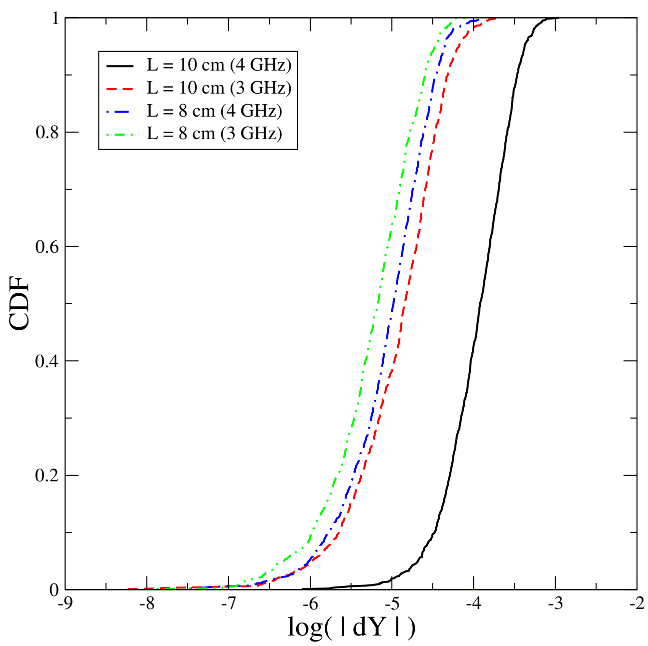

Fig. 3. CDF of the fluctuating part of the cavity admittance: $s = 1$, $s^{'} = 1$.

specifically 600 eigenvalues calculated from a random matrix of the GOE and a loss factor $\alpha = 6$ have been used. Cumulative distribution functions (CDFs) of $\left|\delta Y^{cav}_{ss^{'}}\right|$ have been obtained for different working frequencies, aperture dimensions, and for two selected modes: the diagonal entries $(s = 0, s^{'} = 0)$ Fig. 2, and $(s = 1, s^{'} = 1)$ in Fig. 3. Each simulation has been performed assuming a square $L \times L$ aperture, and generating 1000 random realizations of the cavity. The computational burden due to the Montecarlo generation and the double integral performed for 50 aperture modes required the usage of a parallel supercomputer (total computation time: about 10 hours). We notice a clear departure of distributions corresponding to propagating modes, where admittance increases dramatically for apertures $L/\lambda \gg 1$.

IV. Conclusion

The Random Coupling Model, originally derived for quasi-planar cavities, has been extended to arbitrary three- dimensional cavities. The form of the model, in particular, the relation between the radiation impedances of the ports and the statistically fluctuating matrix generated from Random Matrix Theory is preserved. Methods for calculating the radiation impedance matrices for ports that are treated as superpositions of current distributions or apertures have been described. The resulting expressions should be useful in the theoretical evaluation and prediction of coupled field levels in complex electromagnetic environments, such as reverberation chambers.

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